כבידה 1

2022/23 סמסטר א'

<u>פתרון תרגיל 8 – מסלולים בגיאומטריית שוורצשילד</u>

1. פיזור מכוכב

Solution: Two ways to arrive at the answer are presented.
Version one:
At the radius R of closest approach, the Schwarzschild coordinate components of the four-velocity are ($\theta = \pi/2$):

$$\frac{dr}{d\tau} = 0 \, , \; \frac{d\phi}{d\tau} = \frac{\ell}{R^2} \tag{1}$$

We can find the speed V from the energy measured by the stationary observer, which is

$$\frac{m}{\sqrt{1-V^2}} = E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} \tag{2}$$

where $\mathbf{p} = m\mathbf{u}$ is the four-momentum of the comet and $\mathbf{u}_{\text{obs}}$ is the four-velocity of the stationary observer

$$\mathbf{u}_{\text{obs}} = \left(\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0 \right)$$

Thus,

$$E = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} m \, u^t \tag{3}$$

u^t can be calculated from $\mathbf{u} \cdot \mathbf{u} = -1$ and (1) with the result

$$u^t = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \left(\frac{\ell^2}{R^2} + 1\right)^{1/2}$$

Combining (1), (2), and (3) we have

$$V = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}} \tag{4}$$

It remains to connect ℓ to b and R.

At large r, $\phi \approx b/r$, where b is the impact parameter,

$$\ell \equiv r^2 \frac{d\phi}{d\tau} = -b\left(\frac{dr}{d\tau}\right) = +b\sqrt{e^2 - 1} \tag{5}$$

because $dr/d\tau$ is negative. The energy per unit mass e is determined by the turning point condition

$$e^2 = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) \ . \tag{6}$$

Solving (5) and (6) for ℓ and inserting (4) gives the speed in terms of R and b:

Errata: replace

after "Solving (5) and (6)" with:

$$V = \left(\frac{2M}{R} \frac{1}{1 - R^2/b^2}\right)^{1/2} \ .$$

This is coincidently the same as the formula for V in Newtonian theory.

You can also derive the same relation for V by starting, not from (2), but from the relation that at the turning point $V = u^{\hat{\phi}}/u^{\hat{t}}$ in the orthonormal basis associated with the stationary observer.

2. בליעת פוטונים

Solution: From Figure 9.8 we learn that wherever $1/b^2$ is greater than the height of the barrier the photon will be captured. This is the condition that the impact parameter be less than $\sqrt{27}\,M$ photon will be captured. The cross section is

$$\sigma = \pi(\sqrt{27}\,M)^2 = 27\pi M^2 .$$

3. יציבות של מסלול מעגלי

Solution: In the neighborhood of its maximum at radius r_{max}, the effective potential $V_{\text{eff}}(r)$ behaves

$$V_{\text{eff}}(r) = V_{\text{eff}}(r_{\text{max}}) + \frac{1}{2}\left(\frac{d^2V_{\text{eff}}}{dr^2}\right)_{r_{\text{max}}} (\delta r)^2 + \cdots$$

where $\delta r \equiv r - r_{\text{max}}$. Denote $(d^2V_{\text{eff}}/dr^2)_{r_{\text{max}}}$ by $-K^2$ since it is negative at the maximum of the potential. Eq. (9.29) becomes

$$\frac{1}{2}\left(\frac{d(\delta r)}{d\tau}\right)^2 - \frac{1}{2}K^2(\delta r)^2 = 0 .$$

There are growing and decaying solutions to this upside down harmonic oscillation. The growing solution behaves as

$$\delta r \propto \exp(K\tau) .$$

The time constant τ_* is thus $1/K$. Carrying out the derivatives explicitly, one finds

$$\tau_* = \frac{1}{K} = \sqrt{\frac{r_{\text{max}}^5}{2\ell^2(r_{\text{max}} - 6M)}}$$

where r_{max} is given by (9.34). This diverges as r approaches $6M$ because the stable and unstable circular orbits are coalescing.

4. קואורדינטות אדינגטון – פינקלשטיין

$$t = v - r - 2Mln\left|\frac{r}{2M} - 1\right| \tag{1}$$

For $\frac{r}{2M} - 1 > 0$

$$\begin{aligned} dt &= dv - dr - \left(\frac{r}{2M} - 1\right)^{-1} dr \\ &= dv - \left(1 + \left(\frac{r}{2M} - 1\right)^{-1}\right) dr \\ &= dv - \left(1 + \frac{2M}{r - 2M}\right) dr = dv - \left(\frac{r}{r - 2M}\right) dr \\ &= dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \end{aligned} \tag{2}$$

$$dt^2 = dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dvdr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \tag{3}$$

$\Rightarrow$

$$-\left(1 - \frac{2M}{r}\right) dt^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{4}$$

Plug in Schwarzschild metric in Schwarzschild coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6}$$

The same hold for $\frac{r}{2M} - 1 < 0$ since $\frac{d}{dr} ln\left(Af\left(r\right)\right) = \frac{d}{dr} ln\left(Bf\left(r\right)\right) = f^{-1}\left(r\right) f'\left(r\right)$ for any constants A, B.

5. זמן מקסימלי לפני פגיעה בסינגולריות

Solution: The path of longest proper time should be a geodesic from $r = 2M$ to $r = 0$. Using (9.26) we have for the elapsed proper time

$$\begin{aligned}\tau &= -\int_{2M}^{0} dr/(dr/d\tau) = -\int_{2M}^{0} dr \left[e^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}} \\ &= \int_{0}^{2M} dr \left[e^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(\frac{2M}{r}-1\right)\right]^{-\frac{1}{2}} .\end{aligned}$$

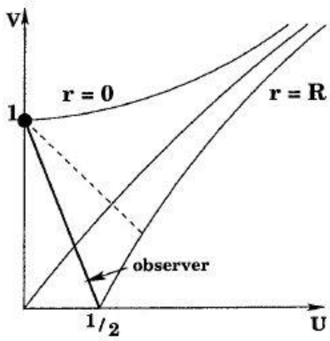
Written this way it is clear that non-zero values of ℓ and e only decrease the proper time from $\ell = e = 0$. That geodesic therefore gives the longest time

$$\tau = \int_0^{2M} dr \left(\frac{2M}{r}-1\right)^{-\frac{1}{2}} = 2M \int_0^1 d\xi \frac{\xi^{\frac{1}{2}}}{\sqrt{1-\xi}} = \pi M .$$

One of the author's students characterized this result as "The more you struggle, the shorter your life."

6. Fools rush in where wise men never go

a)



b) The straight line has a slope of 2 which means the observer is within the 45° lines which are the light cones.

c) The latest time is the value of t at which the 45° dotted line from $U = 0$, $V = 1$ intersects the curve $r = R$. The equation of the 45° line is $V = 1 - U$ so,

$$\left(\frac{R}{2M}-1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \sinh\left(\frac{t}{4M}\right) = 1 - \left(\frac{R}{2M}-1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \cosh\left(\frac{t}{4M}\right)$$

or

$$\frac{1}{2}\sinh\left(\frac{t}{4M}\right) = 1 - \frac{1}{2}\cosh\left(\frac{t}{4M}\right)$$

so that

$$t = 4M \log(2) .$$