Exercise 10 - Solution

Curvature

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Gravity 1 2022-2023

Contents

1 Riemann And Ricci Tensors Identities

1.1 Riemann Tensor In Local Inertial Coordinates

In local inertial coordinates, at point x

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \qquad \partial_\rho g_{\mu\nu}(x) = 0 \qquad \Gamma^\rho_{\mu\nu}(x) = 0 \qquad \partial_\rho \partial_\sigma g_{\mu\nu} \neq 0 \tag{1}$$

thus Riemann tensor contains only derivatives of Γ

$$\begin{aligned}
R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu} &= g_{\rho\lambda}\left(\partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu}\right) \\
&= g_{\rho\lambda}\partial_\mu\left(\frac{1}{2}g^{\lambda\alpha}\left(\partial_\sigma g_{\nu\alpha} + \partial_\nu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\nu}\right)\right) - g_{\rho\lambda}\partial_\nu\left(\frac{1}{2}g^{\lambda\alpha}\left(\partial_\sigma g_{\mu\alpha} + \partial_\mu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\mu}\right)\right) \\
&= \frac{1}{2}g_{\rho\lambda}g^{\lambda\alpha}\partial_\mu\left(\partial_\sigma g_{\nu\alpha} + \partial_\nu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\nu}\right) - \frac{1}{2}g_{\rho\lambda}g^{\lambda\alpha}\partial_\nu\left(\partial_\sigma g_{\mu\alpha} + \partial_\mu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\mu}\right) \\
&= \frac{1}{2}\delta^\alpha_\rho\left[\left(\partial_\mu\partial_\sigma g_{\nu\alpha} + \cancel{\partial_\mu\partial_\nu g_{\sigma\alpha}} - \partial_\mu\partial_\alpha g_{\sigma\nu}\right) - \left(\partial_\nu\partial_\sigma g_{\mu\alpha} + \cancel{\partial_\nu\partial_\mu g_{\sigma\alpha}} - \partial_\nu\partial_\alpha g_{\sigma\mu}\right)\right] \\
&= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right)
\end{aligned} \tag{2}$$

$$\boxed{R_{\rho\sigma\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right)}$$

The following tensor identities we can prove in these coordinates, and the equations will hold in any coordinates.

1.2 Antisymmetry In First Two Indices

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
&= -\frac{1}{2}\left(-\partial_\mu\partial_\sigma g_{\nu\rho} + \partial_\mu\partial_\rho g_{\sigma\nu} + \partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
&= -\frac{1}{2}\left(\partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\sigma\mu} + \partial_\nu\partial_\sigma g_{\mu\rho}\right) \\
&= -R_{\sigma\rho\mu\nu}
\end{aligned} \tag{3}$$

1.3 Antisymmetry In Last Two Indices

$$\begin{aligned}
R_{\rho\sigma\mu\nu} =& \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
=& -\frac{1}{2}\left(-\partial_\mu\partial_\sigma g_{\nu\rho} + \partial_\mu\partial_\rho g_{\sigma\nu} + \partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
=& -\frac{1}{2}\left(\partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\nu\partial_\rho g_{\sigma\mu} - \partial_\mu\partial_\sigma g_{\nu\rho} + \partial_\mu\partial_\rho g_{\sigma\nu}\right) \\
=& -R_{\rho\sigma\nu\mu}
\end{aligned} \tag{4}$$

1.4 Symmetry In Exchanging Pairs Of Indices

$$\begin{aligned}
R_{\rho\sigma\mu\nu} =& \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
=& \frac{1}{2}\left(\partial_\rho\partial_\nu g_{\sigma\mu} - \partial_\rho\partial_\mu g_{\nu\sigma} - \partial_\sigma\partial_\nu g_{\rho\mu} + \partial_\sigma\partial_\mu g_{\nu\rho}\right) \\
=& R_{\mu\nu\rho\sigma}
\end{aligned} \tag{5}$$

1.5 Cyclic Identity

This is unfortunately also called “first Bianchi identity”.

$$\begin{aligned}
R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} =& \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
&+ \frac{1}{2}\left(\partial_\nu\partial_\mu g_{\rho\sigma} - \partial_\nu\partial_\rho g_{\sigma\mu} - \partial_\sigma\partial_\mu g_{\nu\rho} + \partial_\rho\partial_\sigma g_{\nu\mu}\right) \\
&+ \frac{1}{2}\left(\partial_\nu\partial_\sigma g_{\rho\mu} - \partial_\rho\partial_\sigma g_{\mu\nu} - \partial_\mu\partial_\nu g_{\rho\sigma} + \partial_\rho\partial_\mu g_{\sigma\nu}\right) \\
=& 0
\end{aligned} \tag{6}$$

1.6 Bianchi Identity

Again, in local inertial coordinates

$$
\begin{aligned}
\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = \partial_\lambda R_{\rho\sigma\mu\nu} + \partial_\sigma R_{\lambda\rho\mu\nu} + \partial_\rho R_{\sigma\lambda\mu\nu} \\
=& \frac{1}{2}\partial_\lambda \left(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
+& \frac{1}{2}\partial_\sigma \left(\partial_\mu\partial_\rho g_{\nu\lambda} - \partial_\mu\partial_\lambda g_{\nu\rho} - \partial_\nu\partial_\rho g_{\lambda\mu} + \partial_\nu\partial_\lambda g_{\rho\mu}\right) \\
+& \frac{1}{2}\partial_\rho \left(\partial_\mu\partial_\lambda g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\lambda} - \partial_\nu\partial_\lambda g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\lambda}\right) \\
=& \frac{1}{2}\left(\partial_\lambda\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\lambda\partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\lambda\partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\lambda\partial_\nu\partial_\rho g_{\sigma\mu}\right) \\
+& \frac{1}{2}\left(\partial_\sigma\partial_\mu\partial_\rho g_{\nu\lambda} - \partial_\sigma\partial_\mu\partial_\lambda g_{\nu\rho} - \partial_\sigma\partial_\nu\partial_\rho g_{\lambda\mu} + \partial_\sigma\partial_\nu\partial_\lambda g_{\rho\mu}\right) \\
+& \frac{1}{2}\left(\partial_\rho\partial_\mu\partial_\lambda g_{\nu\sigma} - \partial_\rho\partial_\mu\partial_\sigma g_{\nu\lambda} - \partial_\rho\partial_\nu\partial_\lambda g_{\mu\sigma} + \partial_\rho\partial_\nu\partial_\sigma g_{\mu\lambda}\right) \\
=& 0
\end{aligned} \tag{7}
$$

1.7 Ricci Tensor Is Symmetric

Prove with the pairwise symmetry of Riemann (5) and the symmetry of the metric

$$R_{\sigma\nu} = g^{\rho\mu} R_{\rho\sigma\mu\nu} = g^{\mu\rho} R_{\mu\nu\rho\sigma} = R_{\nu\sigma} \tag{8}$$

1.8 Contractions Of Riemann

The metric is symmetric, so there are $\binom{4}{2} = 6$ pairs of contractions to consider.

$$g^{\rho\mu} R_{\rho\sigma\mu\nu} = R_{\sigma\nu} \tag{9}$$

$$g^{\rho\sigma} R_{\rho\sigma\mu\nu} = 0 \tag{10}$$

$$g^{\mu\nu} R_{\rho\sigma\mu\nu} = 0 \tag{11}$$

since Riemann is antisymmetric in its first two indices (3) and its last two indices (4), while the metric is symmetric, and contraction of symmetric indices with antisymmetric indices vanish.

$$g^{\sigma\nu} R_{\rho\sigma\mu\nu} = -g^{\sigma\nu} R_{\sigma\rho\mu\nu} = +g^{\sigma\nu} R_{\sigma\rho\nu\mu} = R_{\rho\mu} \tag{12}$$

$$g^{\rho\nu} R_{\rho\sigma\mu\nu} = -g^{\rho\nu} R_{\rho\sigma\nu\mu} = -R_{\sigma\mu} \tag{13}$$

$$g^{\sigma\mu} R_{\rho\sigma\mu\nu} = -g^{\sigma\mu} R_{\sigma\rho\mu\nu} = -R_{\rho\nu} \tag{14}$$

1.9 Contracted Bianchi Identity

We take Bianchi identity (7) and contract it twice with the inverse metric $g^{\rho\mu} g^{\sigma\nu}$. This specific contractions is chosen such that the first term will produce the Ricci scalar. We use metric compatibility $\nabla g = 0$ to insert the metric into the covariant derivative.

$$\begin{aligned} 0 = g^{\rho\mu} g^{\sigma\nu} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} \right) = \nabla_\lambda R - \nabla^\nu R_{\lambda\nu} - \nabla^\mu R_{\lambda\mu} \\ = \nabla^\nu \left(g_{\lambda\nu} R \right) - 2\nabla^\nu R_{\lambda\nu} = \nabla^\nu \left(g_{\lambda\nu} R - 2R_{\lambda\nu} \right) \end{aligned} \tag{15}$$

Multiply by $-\frac{1}{2}$

$$\nabla^\nu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0 \tag{16}$$

2 The Hyperbolic Plane

2.1 The Poincare Half Plane

$$ds^2 = h^2 \left(y \right) \left(dx^2 + dy^2 \right) \tag{17}$$

The metric components are

$$g_{xx} = g_{yy} = h^2 \qquad g_{xy} = g_{yx} = 0 \tag{18}$$

$$g^{xx} = g^{yy} = h^{-2} \qquad g^{xy} = g^{yx} = 0 \tag{19}$$

The six independent Christoffel symbols are

$$\Gamma^x_{xx} = 0 \tag{20}$$

$$\Gamma^x_{yy} = 0 \tag{21}$$

$$\Gamma^x_{xy} = \frac{1}{2} g^{xx} \partial_y g_{xx} = \frac{h'}{h} \tag{22}$$

$$\Gamma^y_{xx} = -\frac{1}{2} g^{yy} \partial_y g_{xx} = -\frac{h'}{h} \tag{23}$$

$$\Gamma^y_{yy} = \frac{1}{2} g^{yy} \partial_y g_{yy} = \frac{h'}{h} \tag{24}$$

$$\Gamma^{y}_{xy} = 0 \tag{25}$$

Riemann tensor:

$$R^{x}{}_{xxy} = R^{y}{}_{yxy} = 0 \tag{26}$$

because the metric is diagonal.

$$\begin{aligned} R^{x}{}_{yxy} &= \partial_x \Gamma^{x}_{yy} - \partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{xk}\Gamma^{k}_{yy} - \Gamma^{x}_{yk}\Gamma^{k}_{yx} \\ &= -\partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{xy}\Gamma^{y}_{yy} - \Gamma^{x}_{yx}\Gamma^{x}_{yx} = -\partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{xy}\left(\Gamma^{y}_{yy} - \Gamma^{x}_{yx}\right) \\ &= -\left(\frac{h'}{h}\right)' + \frac{h'}{h}\left(\frac{h'}{h} - \frac{h'}{h}\right) = -\left(\frac{h'}{h}\right)' \end{aligned} \tag{27}$$

$$\begin{aligned} R^{y}{}_{xyx} &= \partial_y \Gamma^{y}_{xx} - \partial_x \Gamma^{y}_{xy} + \Gamma^{y}_{yk}\Gamma^{k}_{xx} - \Gamma^{y}_{xk}\Gamma^{k}_{xy} \\ &= \partial_y \Gamma^{y}_{xx} + \Gamma^{y}_{yy}\Gamma^{y}_{xx} - \Gamma^{y}_{xx}\Gamma^{x}_{xy} = \partial_y \Gamma^{y}_{xx} + \Gamma^{y}_{xx}\left(\Gamma^{y}_{yy} - \Gamma^{x}_{xy}\right) \\ &= -\left(\frac{h'}{h}\right)' - \frac{h'}{h}\left(\frac{h'}{h} - \frac{h'}{h}\right) = -\left(\frac{h'}{h}\right)' \end{aligned} \tag{28}$$

Ricci tensor:

$$R_{xx} = R^{k}{}_{xkx} = R^{y}{}_{xyx} = -\left(\frac{h'}{h}\right)' \tag{29}$$

$$R_{yy} = R^{k}{}_{yky} = R^{x}{}_{yxy} = -\left(\frac{h'}{h}\right)' \tag{30}$$

$$R_{xy} = 0 \tag{31}$$

Ricci Scalar:

$$R = g^{ij}R_{ij} = g^{xx}R_{xx} + g^{yy}R_{yy} = -2h^{-2}\left(\frac{h'}{h}\right)' \tag{32}$$

The Ricci scalar of the hyperbolic plane is

$$R = -\frac{2}{a^2} \tag{33}$$

Therefore

$$h^{-2}\left(\frac{h'}{h}\right)' = \frac{1}{a^2} \tag{34}$$

Guess the form

$$h(y) = Ay^{\alpha} \tag{35}$$

$$h' = \alpha A y^{\alpha-1} \tag{36}$$

then

$$\frac{h'}{h} = \frac{\alpha}{y} \tag{37}$$

plug in (34)

$$A^{-2}y^{-2\alpha}\left(-\frac{\alpha}{y^2}\right) = \frac{1}{a^2} \tag{38}$$

$$-\alpha A^{-2}y^{-2\alpha-2} = a^{-2} \tag{39}$$

By equating powers we have

$$\alpha = -1 \tag{40}$$

then

$$A = a \tag{41}$$

$$h(y) = \frac{a}{y} \tag{42}$$

$$\boxed{ds^2 = \frac{a^2}{y^2}\left(dx^2 + dy^2\right)} \tag{43}$$

Another way is to calculate only R_{xyxy} and obtain R_{yxyx} from symmetry.

A third short way in two dimensions is to obtain R directly from $R^x{}_{yxy}$. $R = \frac{2R^x{}_{yxy}}{g_{yy}} = -2h^{-2}\left(\frac{h'}{h}\right)'$.

2.2 The Poincare Disc

$$ds^2 = \frac{4a^4}{\left(a^2 - r^2\right)^2}\left(dr^2 + r^2 d\theta^2\right) \tag{44}$$

Denote

$$q(r) = \frac{2a^2}{a^2 - r^2} \tag{45}$$

The metric components are

$$g_{rr} = q^2 \qquad g_{\theta\theta} = q^2 r^2 \qquad g_{r\theta} = 0 \tag{46}$$

$$g^{rr} = q^{-2} \qquad g^{\theta\theta} = q^{-2} r^{-2} \qquad g^{r\theta} = 0 \tag{47}$$

The Christoffel symbols are

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \partial_r g_{rr} = \frac{q'}{q} \tag{48}$$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -r - r^2 \frac{q'}{q} \tag{49}$$

$$\Gamma^r_{r\theta} = 0 \tag{50}$$

$$\Gamma^\theta_{rr} = 0 \tag{51}$$

$$\Gamma^\theta_{\theta\theta} = 0 \tag{52}$$

$$\Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = \frac{1}{r} + \frac{q'}{q} \tag{53}$$

Riemann tensor: This time we shall calculate directly only the component $R_{r\theta r\theta}$

$$\begin{aligned}
R_{r\theta r\theta} = {} & g_{rk} R^k{}_{\theta r\theta} = g_{rr} R^r{}_{\theta r\theta} \\
= {} & g_{rr} \left(\partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^r_{\theta r} + \Gamma^r_{rk} \Gamma^k_{\theta\theta} - \Gamma^r_{\theta k} \Gamma^k_{\theta r} \right) \\
= {} & g_{rr} \left[\partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{rr} \Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta} \Gamma^\theta_{\theta r} \right] = g_{rr} \left[\partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{\theta\theta} \left(\Gamma^r_{rr} - \Gamma^\theta_{\theta r} \right) \right] \\
= {} & q^2 \left[\partial_r \left(-r - r^2 \frac{q'}{q} \right) + \left(-r - r^2 \frac{q'}{q} \right) \left(\frac{q'}{q} - \left(\frac{1}{r} + \frac{q'}{q} \right) \right) \right] \\
= {} & -q^2 \left[r \frac{q'}{q} + r^2 \left(\frac{q'}{q} \right)' \right]
\end{aligned} \tag{54}$$

$$q' = \frac{4a^2 r}{\left(a^2 - r^2\right)^2} \tag{55}$$

$$\frac{q'}{q} = \frac{2r}{a^2 - r^2} \tag{56}$$

$$\left(\frac{q'}{q} \right)' = \frac{2\left(a^2 - r^2\right) + 4r^2}{\left(a^2 - r^2\right)^2} = \frac{2\left(a^2 + r^2\right)}{\left(a^2 - r^2\right)^2} \tag{57}$$

$$\begin{aligned}
R_{r\theta r\theta} = {} & -q^2 \left[\frac{2r^2}{a^2 - r^2} + r^2 \frac{2\left(a^2 + r^2\right)}{\left(a^2 - r^2\right)^2} \right] = -4q^2 r^2 \frac{a^2}{\left(a^2 - r^2\right)^2} \\
= {} & -r^2 q^4 \frac{1}{a^2}
\end{aligned} \tag{58}$$

Ricci Tensor:

$$\begin{aligned} R_{rr} &= g^{ij} R_{irjr} = g^{\theta\theta} R_{\theta r\theta r} = g^{\theta\theta} R_{r\theta r\theta} \\ &= q^{-2} r^{-2} \left[-r^2 q^4 \frac{1}{a^2} \right] = -\frac{q^2}{a^2} \end{aligned} \tag{59}$$

$$\begin{aligned} R_{\theta\theta} &= g^{ij} R_{i\theta j\theta} = g^{rr} R_{r\theta r\theta} \\ &= q^{-2} \left[-r^2 q^4 \frac{1}{a^2} \right] = -r^2 \frac{q^2}{a^2} \end{aligned} \tag{60}$$

$$\begin{aligned} R &= g^{ij} R_{ij} = g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} \\ &= q^{-2} \left(-\frac{q^2}{a^2} \right) + q^{-2} r^{-2} \left(-r^2 \frac{q^2}{a^2} \right) = -\frac{2}{a^2} \end{aligned} \tag{61}$$

Yes, we got the Ricci scalar of the hyperbolic plane.

$$R = -\frac{2}{a^2}$$

Shortcut for two dimensions:

$$R = \frac{2 R_{r\theta r\theta}}{g} = \frac{2 \left(-r^2 q^4 \frac{1}{a^2} \right)}{q^4 r^2} = -\frac{2}{a^2} \tag{62}$$

3 A Uniform Gravitational Field

$$ds^2 = -\left(1 + gx\right)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{63}$$

Calculate and find ${R^\rho}_{\sigma\mu\nu} = 0$.
Rewrite the metric as

$$ds^2 = -\left(\frac{1}{g} + x \right)^2 d\left(gt\right)^2 + d\left(\frac{1}{g} + x \right)^2 + dy^2 + dz^2 \tag{64}$$

$$y' = y \qquad z' = z \tag{65}$$

Recognize flat metric in hyper-polar coordinates of radius $\left(\frac{1}{g} + x \right)$ and angle gt. The Cartesian coordinates (x', t') are

$$x' = \left(\frac{1}{g} + x \right) \cosh\left(gt\right) \tag{66}$$

$$t' = \left(\frac{1}{g} + x\right) \sinh(gt) \tag{67}$$

$$dx' = \cosh(gt)\, dx + (1 + gx) \sinh(gt)\, dt \tag{68}$$

$$dt' = \sinh(gt)\, dx + (1 + gx) \cosh(gt)\, dt \tag{69}$$

Leaving out $dy'^2 + dz'^2$, we have

$$\begin{aligned} ds^2 =& - dt'^2 + dx'^2 = - \sinh^2(gt)\, dx^2 - (1 + gx)^2 \cosh^2(gt)\, dt^2 - 2\sinh(gt)\, dx\, (1 + gx) \cosh(gt)\, dt \\ & + \cosh^2(gt)\, dx^2 + (1 + gx)^2 \sinh^2(gt)\, dt^2 + 2\cosh(gt)\, dx \sinh(gt)\, dt \\ & = - (1 + gx)^2\, dt^2 + dx^2 \end{aligned} \tag{70}$$

4 Geodesic Deviation Equation

4.1 Adding a Tangent Vector

S^ρ satisfies the geodesic deviation equation

$$(\nabla_u \nabla_u S)^\rho = R^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu S^\nu \tag{71}$$

Given a vector S'^ρ

$$S'^\rho = S^\rho + cu^\rho \tag{72}$$

then

$$S^\rho = S'^\rho - cu^\rho \tag{73}$$

Plug in (71)

$$(\nabla_u \nabla_u (S' - cu))^\rho = R^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu (S'^\nu - cu^\nu) \tag{74}$$

$$(\nabla_u \nabla_u S')^\rho - c\, (\nabla_u \nabla_u u)^\rho = R^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu S'^\nu - cR^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu u^\nu \tag{75}$$

but u satisfies is the geodesic equation

$$\nabla_u u = 0 \tag{76}$$

and

$$R^\rho{}_{\sigma\mu\nu} u^\mu u^\nu = 0 \tag{77}$$

because $R^\rho{}_{\sigma\mu\nu}$ is antisymmetric while $u^\mu u^\nu$ is symmetric in $\mu\nu$.

Therefore S' also satisfies the geodesic deviation equation

$$(\nabla_u \nabla_u S')^\rho = R^\rho{}_{\sigma\mu\nu} u^\sigma u^\mu S'^\nu \tag{78}$$

4.2 Tidal Forces Outside Earth

The Newtonian potential outside earth is

$$\phi = -\frac{GM}{r} \tag{79}$$

where $r = \delta_{ij} x^i x^j = x^i x_i$.

The Riemann tensor is

$$R^i{}_{tjt} = \frac{\partial^2 \phi}{\partial x_i \partial x^j} \tag{80}$$

Use relations

$$\frac{\partial}{\partial x^i} r = \frac{\partial}{\partial x^i} \left(x^j x_j\right)^{\frac{1}{2}} = \frac{1}{2} \left(x^j x_j\right)^{-\frac{1}{2}} \left(\frac{\partial x^j}{\partial x^i} x_j + x^j \frac{\partial x_j}{\partial x^i}\right) = \frac{1}{2} \frac{1}{r} 2\delta^j_i x_j = \frac{x_i}{r} \tag{81}$$

$$\frac{\partial}{\partial x^i} \left(\frac{1}{r}\right) = -\frac{1}{r^2} \frac{\partial}{\partial x^i} r = -\frac{x_i}{r^3} \tag{82}$$

$$\frac{\partial \phi}{\partial x^j} = \frac{\partial}{\partial x^j} \left(-\frac{GM}{r}\right) = GM \frac{x_j}{r^3} \tag{83}$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x_i \partial x^j} &= \frac{\partial}{\partial x_i} \left(GM \frac{x_j}{r^3}\right) = GM \left[\left(\frac{\partial}{\partial x_i} x_j\right) \frac{1}{r^3} + x_j \left(\frac{\partial}{\partial x_i} \frac{1}{r^3}\right)\right] \\ &= GM \left[\delta^i_j \frac{1}{r^3} + x_j \left(-3 \frac{1}{r^4}\right) \frac{x^i}{r}\right] = \frac{GM}{r^3} \left[\delta^i_j - 3 \frac{x^i x_j}{r^2}\right] \end{aligned} \tag{84}$$

$$\boxed{R^i{}_{tjt} = \frac{\partial^2 \phi}{\partial x_i \partial x^j} = \frac{GM}{r^3} \left(\delta^i_j - 3 \frac{x^i x_j}{r^2}\right)} \tag{85}$$

At $x^i = (x, y, z) = (0, 0, r)$ the Riemann tensor non-vanishing components are

$$R^x{}_{txt} = \frac{GM}{r^3} \tag{86}$$

$$R^y{}_{tyt} = \frac{GM}{r^3} \tag{87}$$

$$R^z{}_{tzt} = \frac{GM}{r^3} \left(1 - 3 \frac{r^2}{r^2}\right) = -2 \frac{GM}{r^3} \tag{88}$$

The equations of geodesic deviation are

$$\frac{d^2\xi^x}{dt^2} = -R^x{}_{txt}\xi^x = -\frac{GM}{r^3}\xi^x \tag{89}$$

$$\frac{d^2\xi^y}{dt^2} = -R^y{}_{tyt}\xi^y = -\frac{GM}{r^3}\xi^y \tag{90}$$

$$\frac{d^2\xi^z}{dt^2} = -R^z{}_{tzt}\xi^z = +2\frac{GM}{r^3}\xi^z \tag{91}$$

The squeezing in the x and y directions are half the stretching in the z direction. All together the shape is distorted but the volume of the object is conserved. This is ultimately because the Ricci tensor is zero (Laplace equation in vacuum)

$$R_{tt} = R^\mu{}_{t\mu t} = \frac{GM}{r^3}\left(\delta^i_i - 3\frac{x^i x_i}{r^2}\right) = \frac{GM}{r^3}\left(3 - 3\frac{r^2}{r^2}\right) = 0 \tag{92}$$

Or explicitly

$$R_{tt} = R^\mu{}_{t\mu t} = \frac{GM}{r^3} + \frac{GM}{r^3} - 2\frac{GM}{r^3} = 0 \tag{93}$$