Gravity 1 - Tutorial 11

Gravitational Waves

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Contents

1 Linearized Gravity

1.1 The Formulae

Finding exact solutions to Einstein's equations is hard. Nonetheless, if we have such exact solution, we can make a small perturbation around it and approximate the equations to linear order in the perturbation and find an approximate solution. We will make a perturbation around flat space, i.e., the Minkowski metric solution.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where $\eta_{\mu\nu}$ is Minkowski metric in inertial coordinates, and $h_{\mu\nu}(x)$ is the *metric perturbation*, $|h_{\mu\nu}| \ll 1$.

The inverse metric, to first order in h, is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2}$$

where $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. All the raising, lowering, trace, manipulations to first order are done with η.

The Christoffel symbols to first order in h is

$$\left(\Gamma^{\rho}_{\mu\nu}\right)^{(1)} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) \tag{3}$$

The Riemann tensor to first order in h is

$$\left(R_{\rho\sigma\mu\nu}\right)^{(1)} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\nu\partial_\sigma h_{\rho\mu} - \partial_\mu\partial_\rho h_{\sigma\nu} + \partial_\nu\partial_\rho h_{\sigma\mu}\right) \tag{4}$$

The Ricci tensor to first order in h is

$$\begin{aligned}\left(R_{\mu\nu}\right)^{(1)} &= \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\rho}{}_{\nu} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu V_\nu + \partial_\nu V_\mu - \Box h_{\mu\nu}\right)\end{aligned} \tag{5}$$

where

$$V_\mu \equiv \partial_\nu h^{\nu}{}_{\mu} - \frac{1}{2}\partial_\mu h \tag{6}$$

and h is the **trace** of $h_{\mu\nu}$

$$h \equiv \eta^{\mu\nu}h_{\mu\nu} = h^{\nu}{}_{\nu} \tag{7}$$

and $\Box$ to zero order in h is

$$\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{8}$$

The Ricci scalar to first order in h is

$$R^{(1)} = \partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h \tag{9}$$

The Einstein tensor to first order in h is

$$(G_{\mu\nu})^{(1)} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\nu\partial_\sigma h_{\rho\mu} - \partial_\mu\partial_\rho h_{\sigma\nu} + \partial_\nu\partial_\rho h_{\sigma\mu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\,\Box\, h\right) \tag{10}$$

Gauge transformation of $h_{\mu\nu}$ is

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{11}$$

The Lorenz/harmonic gauge is

$$V_\mu \equiv \partial_\nu h^\nu{}_\mu - \frac{1}{2}\partial_\mu h = 0 \tag{12}$$

It is equivalent to a choice of *harmonic coordinates*, i.e., **coordinate functions** with zero Laplacian

$$\Box\, x^\mu = 0 \tag{13}$$

The linearized vacuum Einstein equation is

$$(R_{\mu\nu})^{(1)} = 0 \tag{14}$$

In the Lorenz gauge it is the wave equation

$$\Box\, h_{\mu\nu} = 0 \tag{15}$$

1.2 Gauge Invariance

Exercise: Show explicitly that the linearized Riemann tensor (4) is invariant under gauge transformation (11).

Plug (11) into (4)

$$
\begin{aligned}
\left(R'_{\rho\sigma\mu\nu}\right)^{(1)} = &\frac{1}{2}\left(\partial_\mu\partial_\sigma\left(h_{\rho\nu} - \partial_\rho\xi_\nu - \partial_\nu\xi_\rho\right) - \partial_\nu\partial_\sigma\left(h_{\rho\mu} - \partial_\rho\xi_\mu - \partial_\mu\xi_\rho\right)\right)\\
&-\frac{1}{2}\left(\partial_\mu\partial_\rho\left(h_{\sigma\nu} - \partial_\sigma\xi_\nu - \partial_\nu\xi_\sigma\right) + \partial_\nu\partial_\rho\left(h_{\sigma\mu} - \partial_\sigma\xi_\mu - \partial_\mu\xi_\sigma\right)\right)\\
&=R_{\rho\sigma\mu\nu} + \frac{1}{2}\left(-\partial_\mu\partial_\sigma\partial_\rho\xi_\nu - \partial_\mu\partial_\sigma\partial_\nu\xi_\rho + \partial_\nu\partial_\sigma\partial_\rho\xi_\mu + \partial_\nu\partial_\sigma\partial_\mu\xi_\rho\right)\\
&+\frac{1}{2}\left(\partial_\mu\partial_\rho\partial_\sigma\xi_\nu + \partial_\mu\partial_\rho\partial_\nu\xi_\sigma - \partial_\nu\partial_\rho\partial_\sigma\xi_\mu - \partial_\nu\partial_\rho\partial_\mu\xi_\sigma\right)\\
&=R_{\rho\sigma\mu\nu}
\end{aligned}
\tag{16}
$$

This is akin to gauge transformation of electromagnetic potential $A'_\mu = A_\mu + \partial_\mu\alpha$ that leaves the EM field strength (curvature) tensor invariant

$$
F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu\left(A_\nu + \partial_\nu\alpha\right) - \partial_\nu\left(A_\mu + \partial_\mu\alpha\right) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}
\tag{17}
$$

1.3 Metric Perturbation Example

Exercise: Consider the metric perturbation

$$
h_{\mu\nu} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \sin\left(k\left(x - t\right)\right)
\tag{18}
$$

for arbitrary constants a and c.

1. Show that $h_{\mu\nu}$ solves the linearized Einstein equations in vacuum (14).
2. Find the functions $\xi_\mu(x)$ that transform it into a diagonal form where $c = 0$. What is the new value of a?

1.3.1 Show that $h_{\mu\nu}$ is a solution

Ricci tensor (5) has two kinds of terms, $\Box h_{\mu\nu}$ and V_μ (6).

$$
\begin{aligned}
\Box h_{\mu\nu} = &a_{\mu\nu}\left(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2\right)\sin\left(k\left(x - t\right)\right)\\
&=a_{\mu\nu}\left[-\left(-k^2\sin\left(k\left(x - t\right)\right)\right) + \left(-k^2\sin\left(k\left(x - t\right)\right)\right)\right] = 0
\end{aligned}
\tag{19}
$$

where $a_{\mu\nu}$ is the constant matrix given in (18). The trace $h_{\mu\nu}$ is

$$\begin{aligned} h &= \eta^{\rho\sigma} h_{\rho\sigma} = -h_{tt} + h_{xx} + h_{yy} + h_{zz} \\ &= [-(c) + (-c) + a + (-a)] \sin(k(x-t)) = -2c \sin(k(x-t)) \end{aligned} \tag{20}$$

Notice it is not zero. We need to calculate

$$V_\mu = \partial_\nu h^\nu{}_\mu - \frac{1}{2}\partial_\mu h = \eta^{\nu\rho}\partial_\nu h_{\rho\mu} - \frac{1}{2}\partial_\mu h \tag{21}$$

Since $\eta^{\nu\rho}$ and $h_{\rho\mu}$ are diagonal, we find

$$V_t = \eta^{tt}\partial_t h_{tt} - \frac{1}{2}\partial_t h = \partial_t \left(-h_{tt} - \frac{1}{2}h\right) = \partial_t \left[\left(-c - \frac{1}{2}(-2c)\right)\sin(k(x-t))\right] = 0 \tag{22}$$

$$V_x = \eta^{xx}\partial_x h_{xx} - \frac{1}{2}\partial_x h = \partial_x \left[h_{xx} - \frac{1}{2}h\right] = \partial_x \left[\left((-c) - \frac{1}{2}(-2c)\right)\sin(k(x-t))\right] = 0 \tag{23}$$

$$V_y = 0 \qquad V_z = 0 \tag{24}$$

We found that indeed

$$\left(R_{\mu\nu}\right)^{(1)} = \frac{1}{2}\left(\partial_\mu V_\nu + \partial_\nu V_\mu - \Box h_{\mu\nu}\right) = 0 \tag{25}$$

1.3.2 Find the functions $\xi_\mu(x)$

We demand that the gauge transformation (11) would yield $h'_{tt} = 0$

$$0 \stackrel{!}{=} h'_{tt} = h_{tt} - 2\partial_t \xi_t \tag{26}$$

$\Rightarrow$

$$\partial_t \xi_t = \frac{1}{2}h_{tt} = \frac{c}{2}\sin(k(x-t)) \tag{27}$$

integrate

$$\boxed{\xi_t = \frac{c}{2k}\cos(k(x-t))} \tag{28}$$

We also demand that the gauge transformation (11) would yield $h'_{xx} = 0$

$$0 \stackrel{!}{=} h'_{xx} = h_{xx} - 2\partial_x \xi_x \tag{29}$$

$\Rightarrow$

$$\partial_x \xi_x = \frac{1}{2} h_{xx} = \frac{-c}{2} \sin\left(k\left(x-t\right)\right) \tag{30}$$

integrate

$$\boxed{\xi_x = \frac{c}{2k} \cos\left(k\left(x-t\right)\right)} \tag{31}$$

Check that it does not create non diagonal element h'_{tx}

$$h'_{tx} = h_{tx} - \partial_t \xi_x - \partial_x \xi_t = -\left(\partial_t + \partial_x\right) \frac{c}{2k} \cos\left(k\left(x-t\right)\right) = 0 \tag{32}$$

Take

$$\xi_y = \xi_z = 0 \tag{33}$$

and since ξ_t and ξ_x do not depend on y and z, $\partial_y \xi_\mu = \partial_z \xi_\mu = \partial_\mu \xi_y = \partial_\mu \xi_z = 0$. Therefore no components of $h_{\mu\nu}$ change except for h_{tt} and h_{xx}. "a" does not change.

2 Production Of Gravitational Waves (not for exam)

2.1 Linearized Einstein Equation With Sources

Define the *trace-reversed* perturbation

$$\overline{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \tag{34}$$

Exercise: Justify the name "trace-reversed" for $\overline{h}_{\mu\nu}$.

$$\eta^{\mu\nu} \overline{h}_{\mu\nu} = \eta^{\mu\nu} \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h\right) = h - \frac{1}{2} 4h = -h \tag{35}$$

$\overline{h}_{\mu\nu}$ has the negative trace of $h_{\mu\nu}$ (in four dimensions). In the transverse-traceless gauge they coincide.

Exercise: Write the Lorenz gauge condition (12) for $\overline{h}_{\mu\nu}$.

$$\partial_\nu h^{\nu\mu} - \frac{1}{2} \partial^\mu h = \partial_\nu \left(h^{\nu\mu} - \frac{1}{2} \eta^{\nu\mu} h\right) = \partial_\nu \overline{h}^{\mu\nu} = 0 \tag{36}$$

Exercise: Write the linearized Einstein equations with sources in Lorenz gauge.

In Lorenz gauge the Ricci tensor (5) is

$$(R_{\mu\nu})^{(1)} = -\frac{1}{2}\Box h_{\mu\nu} \tag{37}$$

and the Ricci scalar (9) is

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \partial_\mu\partial^\mu h = \partial_\mu\left(\frac{1}{2}\partial^\mu h\right) - \partial_\mu\partial^\mu h = -\frac{1}{2}\Box h \tag{38}$$

so the Einstein tensor is

$$\begin{aligned}(G_{\mu\nu})^{(1)} &= (R_{\mu\nu})^{(1)} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} \\ &= -\frac{1}{2}\left(\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h\right) = -\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = -\frac{1}{2}\Box\,\overline{h}_{\mu\nu}\end{aligned} \tag{39}$$

The linearized Einstein equations with sources in Lorenz gauge is

$$\Box\,\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{40}$$

Linearized Einstein equation with source in Lorentz gauge

(40) are wave equations with sources for each $\overline{h}_{\mu\nu}$. The vacuum equations have Ricci tensor on the l.f.h, and are wave equations for $h_{\mu\nu}$ in Lorenz gauge (15). The equations with sources in Lorenz gauge turn out to be wave equations for $\overline{h}_{\mu\nu}$ since in the l.h.s there is the Einstein tensor, which is the "trace-reversed Ricci tensor".

2.2 The General Solution

Write down the general solution of (40).

This step is exactly like in electromagnetism, so we quote in short.

The Green function $G(x^\sigma - y^\sigma)$ of the $\Box$ operator is the solution to a point (delta function) source

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) \tag{41}$$

where y^σ is the position of the source and x^σ is where we evaluate the field (in spacetime). The *retarded* solution is

$$G\left(x^{\sigma}-y^{\sigma}\right)=-\frac{1}{4\pi\left|\boldsymbol{x}-\boldsymbol{y}\right|}\delta\left[\left|\boldsymbol{x}-\boldsymbol{y}\right|-\left(x^{0}-y^{0}\right)\right]\theta\left(x^{0}-y^{0}\right) \tag{42}$$

where $\boldsymbol{x},\boldsymbol{y}$ are three-vectors, $x^{0}\equiv t$, θ is Heaviside function. The general solution is an integral over the source with the Green function

$$\bar{h}_{\mu\nu}\left(x^{\sigma}\right)=-16\pi\int G\left(x^{\sigma}-y^{\sigma}\right)T_{\mu\nu}\left(y^{\sigma}\right)d^{4}y \tag{43}$$

(apply $\Box_{x}$ to both sides of (43) and with the Green function definition (41) recover (40)).

We preform the integral in (43) over dt, by the delta function in (42), and get

$$\bar{h}_{\mu\nu}\left(x^{\sigma}\right)=4\int\frac{1}{\left|\boldsymbol{x}-\boldsymbol{y}\right|}T_{\mu\nu}\left(t-\left|\boldsymbol{x}-\boldsymbol{y}\right|,\boldsymbol{y}\right)d^{3}y \tag{44}$$

where we substituted $y^{0}=x^{0}-\left|\boldsymbol{x}-\boldsymbol{y}\right|=t-\left|\boldsymbol{x}-\boldsymbol{y}\right|$. The retarded time is

$$t_{r}=t-\left|\boldsymbol{x}-\boldsymbol{y}\right| \tag{45}$$

since $\left|\boldsymbol{x}-\boldsymbol{y}\right|$ times $c=1$ is the time it takes the signal from the source at $\boldsymbol{y}$ to reach $\boldsymbol{x}$. The general solution can be written as

$$\bar{h}_{\mu\nu}\left(t,\boldsymbol{x}\right)=4\int\frac{1}{\left|\boldsymbol{x}-\boldsymbol{y}\right|}T_{\mu\nu}\left(t_{r},\boldsymbol{y}\right)d^{3}y \tag{46}$$

General solution to wave equations with source

2.3 An Approximate Solution

Use a time-Fourier transform and approximate (46) for a source which is isolated, far away, and slowly moving (non-relativistic).

First we make a time - Fourier transform to (46)

$$
\begin{aligned}
\widetilde{\overline{h}}_{\mu\nu}(\omega,\boldsymbol{x}) =& \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\overline{h}_{\mu\nu}(t,\boldsymbol{x}) \\
=& \frac{4}{\sqrt{2\pi}}\int dt d^3y e^{i\omega t}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}T_{\mu\nu}(t-|\boldsymbol{x}-\boldsymbol{y}|,\boldsymbol{y}) \\
=& \frac{4}{\sqrt{2\pi}}\int dt_r d^3y e^{i\omega t_r}\frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}T_{\mu\nu}(t_r,\boldsymbol{y}) \\
=& 4\int d^3y \frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}\widetilde{T}_{\mu\nu}(\omega,\boldsymbol{y}) \qquad (47)
\end{aligned}
$$

Now we make the approximations, which mean that we consider a source centered at spatial distance r with, which is confined to a radius $\delta r \ll r$. Slowly moving source would emit low frequencies. $\delta r \ll \omega^{-1} \sim \lambda$ (long wavelength approximation). So, $|\boldsymbol{x}-\boldsymbol{y}| \sim r$ and $\omega|\boldsymbol{x}-\boldsymbol{y}| \sim \omega r$, and we can replace $\frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}$ by $\frac{e^{i\omega r}}{r}$ and take it out of the integral.

$$
\widetilde{\overline{h}}_{\mu\nu}(\omega,\boldsymbol{x}) = 4\frac{e^{i\omega r}}{r}\int d^3y \widetilde{T}_{\mu\nu}(\omega,\boldsymbol{y}) \qquad (48)
$$

Transform back to

$$
\overline{h}_{\mu\nu}(t,\boldsymbol{x}) = \frac{4}{r}\int d^3y T_{\mu\nu}(t_r,\boldsymbol{y}) \qquad (49)
$$

Weak, isolated, far and slow source solution

where now

$$
t_r = t - r \qquad (50)
$$

2.4 Tensor Virial Theorem

Exercise 1: Show that in flat spacetime $\partial_i\partial_j T^{ij} = \partial_t^2 T^{tt}$.

Energy-momentum conservation in flat spacetime is

$$
\partial_\mu T^{\mu\nu} = 0 \qquad (51)
$$

The t and i components explicitly are

$$
\partial_t T^{tt} + \partial_i T^{it} = 0 \qquad (52)
$$

$$
\partial_t T^{ti} + \partial_j T^{ji} = 0 \qquad (53)
$$

Using (53), commute partial derivatives, and then (52), yield

$$\partial_i \partial_j T^{ij} = -\partial_i \partial_t T^{ti} = -\partial_t \partial_i T^{ti} = \partial_t \partial_t T^{tt} \tag{54}$$

Exercise 2: Show that

$$\int d^3 y T^{ij} = \frac{1}{2} \frac{d^2}{dt^2} \int d^3 y T^{tt} y^i y^j \tag{55}$$

We multiply (54) by $y^i y^j$ and integrate over $\int d^3 y$. Use integration by parts, and that the source is isolated so the surface terms vanish, the l.h.s is

$$\begin{aligned}\int dy^3 y^i y^j \partial_k \partial_l T^{kl} &= -\int dy^3 \partial_k \left(y^i y^j\right) \partial_l T^{kl} = -\int dy^3 \left(\delta^i_k y^j + y^i \delta^j_k\right) \partial_l T^{kl} \\ &= -\int dy^3 \left(y^j \partial_l T^{il} + y^i \partial_l T^{jl}\right) = \int dy^3 \left(\left(\partial_l y^j\right) T^{il} + \left(\partial_l y^i\right) T^{jl}\right) \\ &= \int dy^3 \left(\delta^j_l T^{il} + \delta^i_l T^{jl}\right) = \int dy^3 \left(T^{ij} + T^{ji}\right) = 2 \int dy^3 T^{ij}\end{aligned} \tag{56}$$

Together with the r.h.s, (54) becomes

$$\int dy^3 T^{ij} = \frac{1}{2} \frac{d^2}{dt^2} \int dy^3 T^{tt} y^i y^j \tag{57}$$

Tensor virial theorem

2.5 The Quadrupole Formula

Define the quadrupole moment tensor of the energy density of the source

$$I_{ij}(t) = \int dy^3 T^{tt} y_i y_j \tag{58}$$

Quadrupole moment tensor

With (58), (57) becomes

$$\int dy^3 T^{ij} = \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} \tag{59}$$

and the approximate solution to the spatial components of the metric perturbation (49) becomes

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{2}{r} \frac{d^2 I_{ij}}{dt^2} (t_r) \tag{60}$$

The quadrupole formula

What about the remaining components of the metric perturbation, $h^{t\mu}$? They can be deduced from h^{ij} and the Lorenz gauge condition.

3 Gravitational Radiation From Binary Stars

3.1 Metric Perturbation

Exercise: Consider a simple binary star (two stars orbiting each other). It consists of two stars of mass M in a circular orbit in the $x^1 - x^2$ plane, at distance R from their common center of mass. Calculate $\overline{h}_{ij}(t, \boldsymbol{x})$ in the weak, isolated, far, slow approximation, and for Newtonian orbit of the stars.

The Newton second law for a mass M in a circular orbit of radius R, under the gravitational force of another mass M at distance $2R$, is

$$\frac{M^2}{(2R)^2} = M\Omega^2 R \tag{61}$$

where Ω is the **orbital** frequency (velocity). It reads

$$\Omega = \left(\frac{M}{4R^3}\right)^{\frac{1}{2}} \tag{62}$$

The positions of star a is

$$x_a^1 = R\cos(\Omega t) \tag{63}$$
$$x_a^2 = R\sin(\Omega t) \tag{64}$$
$$x_a^3 = 0 \tag{65}$$

The position of star b is the negative coordinates

$$x_b^1 = -R\cos(\Omega t) \tag{66}$$
$$x_b^2 = -R\sin(\Omega t) \tag{67}$$
$$x_b^3 = 0 \tag{68}$$

The quadrupole moment for this discrete system cam be written as

$$I_{ij} = \sum_c m_c x_c^i x_c^j = M x_a^i x_a^j + M x_b^i x_b^j = M x_a^i x_a^j + M \left(-x_a^i\right)\left(-x_a^j\right)$$
$$= 2M x_a^i x_a^j \tag{69}$$

The components of the quadrupole moment tensor are

$$I_{11} = 2MR^2 \cos^2(\Omega t) = MR^2 \left(1 + \cos(2\Omega t)\right) \tag{70}$$

$$I_{22} = 2MR^2 \sin^2(\Omega t) = MR^2 \left(1 - \cos(2\Omega t)\right) \tag{71}$$

$$I_{12} = 2MR^2 \sin(\Omega t)\cos(\Omega t) = MR^2 \sin(2\Omega t) \tag{72}$$

$$I_{13} = I_{23} = I_{33} = 0 \tag{73}$$

Differentiate (70),(71)(72) twice, and plug into (60)

$$\overline{h}_{ij}(t, \boldsymbol{x}) = \frac{8}{r} M\Omega^2 R^2 \begin{pmatrix} -\cos(2\Omega t_r) & -\sin(2\Omega t_r) & 0 \\ -\sin(2\Omega t_r) & \cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{74}$$

The radiation frequency $\omega = 2\Omega$ is twice the orbital frequency, since the total binary star reach the same configuration after half of the period of its stars.

3.2 Energy Loss Due To Gravitational Radiation

Define a *reduced/traceless quadrupole moment* as

$$J_{ij} = I_{ij} - \frac{1}{3} I \delta_{ij} \tag{75}$$

where I is the trace of I_{ij}

$$I \equiv \delta^{kl} I_{kl} \tag{76}$$

Explicitly,

$$J_{ij} = \int d^3y T^{tt} \left(y_i y_j - \frac{1}{3} y^2 \delta_{ij}\right) \tag{77}$$

This is the actual quadrupole moment that appears in a multipole expansion of Newtonian potential.

The power radiated in gravitational waves is

$$P = -\frac{1}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle \tag{78}$$

Traceless quadrupole formula for radiated power

where the quadrupole moment is evaluated at the retarded time $t-r$, and the $\langle ... \rangle$ denotes the time average over a period. We will not derive this formula, yet here are few remarks on some logic in this formula: Energy is proportional to amplitude squared of a wave; Power is energy/time and is dimensionless in geometrical units, and so is $\frac{d^3 J_{ij}}{dt^3}$; A spherically symmetric source emits no radiation, and likewise it has only monopole in the multipole expansion, so $J_{ij} = 0$ (unlike I_{ij}); The total power is a scalar w.r.t space rotations.

Exercise: Calculate the power emitted in radiation by a binary star.

First we compute J_{ij} for (70)-(73). The trace I is

$$I = I_{11} + I_{22} + I_{33} = 2MR^2 \tag{79}$$

Plug in (75)

$$J_{ij} = \frac{MR^2}{3}\begin{pmatrix} 1+3\cos(2\Omega t) & 3\sin(2\Omega t) & 0 \\ 3\sin(2\Omega t) & 1-3\cos(2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{80}$$

$$\frac{d^3 J_{ij}}{dt^3} = 8MR^2\Omega^3\begin{pmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{81}$$

by matrix multiplication and trace

$$\frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3} = \left(8MR^2\Omega^3\right)^2 Tr \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 128M^2R^4\Omega^6 \tag{82}$$

No time average is needed in this case, plug in (78)

$$P = -\frac{128}{5}M^2R^4\Omega^6 \tag{83}$$

Power emitted in radiation form a binary star

and only in terms of M, R, plug Ω (62)

$$P = -\frac{2}{5}\left(\frac{M}{R}\right)^5 \tag{84}$$