

Exercise 11 - Solution

Gravitational Waves

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1 Linearized Gravity

1.1 Inverse Metric

The Linearized metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (1)$$

The alleged linearized inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad (2)$$

where

$$h^{\mu\nu} \equiv \eta^{\rho\mu}\eta^{\sigma\nu}h_{\rho\sigma} \quad (3)$$

Multiply the metric (1) and its presumed inverse (2)

$$\begin{aligned} g_{\mu\sigma}g^{\sigma\nu} &= (\eta_{\mu\sigma} + h_{\mu\sigma})(\eta^{\sigma\nu} - h^{\sigma\nu}) \\ &= \eta_{\mu\sigma}\eta^{\sigma\nu} - \eta_{\mu\sigma}h^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} - h_{\mu\sigma}h^{\sigma\nu} \\ &= \delta_{\mu}^{\nu} + O(h^2) \end{aligned} \quad (4)$$

To first order we got the unit matrix, so indeed (2) is the linearized inverse.

1.2 Linearized Christoffel And Curvature Tensors

Christoffel symbol

$$\begin{aligned} (\Gamma_{\mu\nu}^{\rho})^{(1)} &= \frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})(\partial_{\mu}(\eta_{\nu\sigma} + h_{\nu\sigma}) + \partial_{\nu}(\eta_{\mu\sigma} + h_{\mu\sigma}) - \partial_{\sigma}(\eta_{\mu\nu} + h_{\mu\nu})) \\ &= \frac{1}{2}\eta^{\rho\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}) \end{aligned} \quad (5)$$

since $\partial_{\mu}\eta_{\nu\sigma} = 0$ and $h^{\rho\sigma}\partial_{\mu}h_{\nu\sigma} \sim O(2)$.

Riemann tensor

$$\begin{aligned} (R_{\rho\sigma\mu\nu})^{(1)} &= (\eta_{\rho\lambda} + h_{\rho\lambda})\left(\partial_{\mu}(\Gamma_{\sigma\nu}^{\lambda})^{(1)} - \partial_{\nu}(\Gamma_{\sigma\mu}^{\lambda})^{(1)} + (\Gamma_{\mu\alpha}^{\lambda})^{(1)}(\Gamma_{\sigma\nu}^{\alpha})^{(1)} - (\Gamma_{\nu\alpha}^{\lambda})^{(1)}(\Gamma_{\sigma\mu}^{\alpha})^{(1)}\right) \\ &= \eta_{\rho\lambda}\left(\partial_{\mu}(\Gamma_{\sigma\nu}^{\lambda})^{(1)} - \partial_{\nu}(\Gamma_{\sigma\mu}^{\lambda})^{(1)}\right) \\ &= \frac{1}{2}(\partial_{\mu}\partial_{\sigma}h_{\rho\nu} - \partial_{\nu}\partial_{\sigma}h_{\rho\mu} - \partial_{\mu}\partial_{\rho}h_{\sigma\nu} + \partial_{\nu}\partial_{\rho}h_{\sigma\mu}) \end{aligned} \quad (6)$$

Ricci tensor

$$\begin{aligned} (R_{\sigma\nu})^{(1)} &= (\eta^{\rho\mu} - h^{\rho\mu})(R_{\rho\sigma\mu\nu})^{(1)} = \eta^{\rho\mu}(R_{\rho\sigma\mu\nu})^{(1)} \\ &= \frac{1}{2}(\partial_\mu\partial_\sigma h^\mu{}_\nu + \partial_\nu\partial_\mu h^\mu{}_\sigma - \partial_\nu\partial_\sigma h - \square h_{\sigma\nu}) \end{aligned} \quad (7)$$

Ricci scalar

$$\begin{aligned} R^{(1)} &= \eta^{\sigma\nu}(R_{\sigma\nu})^{(1)} = \frac{1}{2}(\partial_\mu\partial_\nu h^{\mu\nu} + \partial_\nu\partial_\mu h^{\mu\nu} - \partial_\nu\partial^\nu h - \square h^\nu{}_\nu) \\ &= \partial_\mu\partial_\nu h^{\mu\nu} - \square h \end{aligned} \quad (8)$$

Einstein tensor

$$\begin{aligned} (G_{\mu\nu})^{(1)} &= (R_{\mu\nu})^{(1)} - \frac{1}{2}R^{(1)}\eta_{\mu\nu} \\ &= \frac{1}{2}(\partial_\sigma\partial_\mu h^\sigma{}_\nu + \partial_\nu\partial_\sigma h^\sigma{}_\mu - \partial_\mu\partial_\nu h - \square h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\square h) \end{aligned} \quad (9)$$

1.3 Harmonic Gauge

Harmonic coordinates are a set of coordinates such that each of them is a harmonic **function**, i.e., satisfies the Laplace equation

$$\square x^\mu = 0 \quad (10)$$

These are four equations, where the \square is acting on the functions x^1, x^2, x^3, x^4 . The index μ is part of the name of the function under consideration.

First we use the definition $\square = g^{\rho\sigma}\nabla_\rho\nabla_\sigma$, and that the covariant derivative reduces to the partial derivative when acting on a function

$$\square x^\mu = g^{\rho\sigma}\nabla_\rho\nabla_\sigma x^\mu = g^{\rho\sigma}\nabla_\rho(\partial_\sigma x^\mu) = g^{\rho\sigma}\nabla_\rho\delta_\sigma^\mu \quad (11)$$

Now, the second covariant derivative is acting on an object with one lower index σ (again, the μ is only part of the name of the coordinate function x^μ)

$$\square x^\mu = g^{\rho\sigma}\nabla_\rho\delta_\sigma^\mu = g^{\rho\sigma}\partial_\rho\delta_\sigma^\mu - g^{\rho\sigma}\Gamma_{\rho\sigma}^\nu\delta_\nu^\mu = 0 - g^{\rho\sigma}\Gamma_{\rho\sigma}^\mu \quad (12)$$

So the harmonic condition (10) is equivalent to

$$\Gamma_{\rho\sigma}^\mu g^{\rho\sigma} = 0 \quad (13)$$

A second way to write the d'Alembertian is terms of the metric, $\square = \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\sigma} \partial_\sigma \right)$

$$\begin{aligned} \square x^\mu &= \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\sigma} \partial_\sigma x^\mu \right) = \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\sigma} \delta_\sigma^\mu \right) \\ &= \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\mu} \right) \end{aligned} \quad (14)$$

So the harmonic condition (10) is equivalent to

$$\partial_\nu \left(\sqrt{|g|} g^{\nu\mu} \right) = 0 \quad (15)$$

Let us derive the linearized harmonic condition from both expressions. From (13) we have

$$\begin{aligned} 0 &= \Gamma_{\mu\nu}^\rho g^{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) g^{\mu\nu} \\ &= g^{\rho\sigma} g^{\mu\nu} \partial_\mu g_{\nu\sigma} - \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} \partial_\sigma g_{\mu\nu} \\ &\approx \eta^{\sigma\rho} \eta^{\mu\nu} \partial_\mu h_{\nu\sigma} - \frac{1}{2} \eta^{\sigma\rho} \eta^{\mu\nu} \partial_\sigma h_{\mu\nu} \\ &= \partial_\mu h^{\mu\rho} - \frac{1}{2} \eta^{\sigma\rho} \partial_\sigma h \end{aligned} \quad (16)$$

\Rightarrow

$$\partial_\nu h^{\nu\mu} - \frac{1}{2} \eta^{\mu\nu} \partial_\nu h = 0 \quad (17)$$

where we noticed that the first two terms from Γ are identical because of the contractions.

From (15) we have

$$\begin{aligned} 0 &= \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\mu} \right) = \left(\frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} \right) g^{\nu\mu} + \partial_\nu g^{\nu\mu} \\ &= \left(\frac{1}{2} g^{\rho\sigma} \partial_\nu g_{\rho\sigma} \right) g^{\nu\mu} + \partial_\nu g^{\nu\mu} \\ &\approx \frac{1}{2} \eta^{\rho\sigma} \partial_\nu h_{\rho\sigma} \eta^{\nu\mu} - \partial_\nu h^{\nu\mu} \\ &= \frac{1}{2} \eta^{\nu\mu} \partial_\nu h - \partial_\nu h^{\nu\mu} \end{aligned} \quad (18)$$

\Rightarrow

$$\partial_\nu h^{\nu\mu} - \frac{1}{2}\eta^{\mu\nu}\partial_\nu h = 0 \quad (19)$$

where in the step to the second line we used the identity

$$\frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|} = \frac{1}{2}g^{\rho\sigma}\partial_\nu g_{\rho\sigma} \quad (20)$$

(see recitation 9 section 2.3.2), and the minus sign in the third line is from the inverse metric (2).

For new coordinates $x'^\mu = x^\mu + \xi^\mu$ to be harmonic as well, we require

$$0 = \square x'^\mu = \square(x^\mu + \xi^\mu) = \square x^\mu + \square \xi^\mu = 0 + \square \xi^\mu \quad (21)$$

$$\square \xi^\mu = 0 \quad (22)$$

The transformation functions ξ^μ need to be harmonic.

1.4 Linearized Vacuum Einstein Equation In Harmonic Gauge

Linearized Vacuum Einstein equation is

$$R_{\mu\nu}^{(1)} = 0 \quad (23)$$

From (7) it is

$$\frac{1}{2}(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\mu\partial_\rho h^\rho{}_\nu - \partial_\mu\partial_\nu h - \square h_{\mu\nu}) = 0 \quad (24)$$

Harmonic gauge (19) can be written with a lower index as

$$\partial_\nu h^\nu{}_\mu = \frac{1}{2}\partial_\mu h \quad (25)$$

Plug in (24)

$$\frac{1}{2}\left(\frac{1}{2}\partial_\nu\partial_\mu h + \frac{1}{2}\partial_\mu\partial_\nu h - \partial_\mu\partial_\nu h - \square h_{\mu\nu}\right) = -\frac{1}{2}\square h_{\mu\nu} = 0 \quad (26)$$

Therefore

$$\square h_{\mu\nu} = 0$$

1.5 Linearized Einstein Equation With Sources In Harmonic Gauge

Trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad (27)$$

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h \quad (28)$$

Condition (19) is

$$\partial_\nu \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h \right) = 0 \quad (29)$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0$$

To write Einstein equation with sources we need the Linearized Einstein Tensor in harmonic gauge. We already found Ricci tensor in this gauge, namely $(R_{\mu\nu})^{(1)} = -\frac{1}{2}\square h_{\mu\nu}$. Now we find the linearized Ricci scalar (8) in this gauge

$$R^{(1)} = \partial_\mu \partial_\nu h^{\mu\nu} - \square h = \frac{1}{2}\partial_\mu \eta^{\mu\nu} \partial_\nu h - \square h = -\frac{1}{2}\square h \quad (30)$$

The linearized Einstein tensor in harmonic gauge is

$$\begin{aligned} (G_{\mu\nu})^{(1)} &= (R_{\mu\nu})^{(1)} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} \\ &= -\frac{1}{2}\square h_{\mu\nu} + \frac{1}{2}\frac{1}{2}\eta_{\mu\nu}\square h \\ &= -\frac{1}{2}\square \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \right) \\ &= -\frac{1}{2}\square \bar{h}_{\mu\nu} \end{aligned} \quad (31)$$

The Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ becomes

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad (32)$$

2 Production Of Gravitational Waves By A Rotating Rod

We need to calculate the quadrupole moment of the rotating rod. One way is to do it in the rotating frame of the rod (x', y', z') . The center of mass is at the origin, and we align the rod along the x' -axes. It has a constant linear density $\rho = \frac{M}{L}$ between $-\frac{L}{2} < x' < \frac{L}{2}$. There is no mass in the y' and z' directions, so the only non-zero component of the quadrupole tensor is

$$I^{x'x'} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} x'^2 dx' = \frac{1}{12} ML^2 \quad (33)$$

This is just the moment of inertia of a uniform rod about its center of mass.

In order to use the quadrupole formula we need to transform I^{ij} back to the inertial frame (x, y, z) , where the rod is rotating with angular velocity Ω . Suppose it rotates counter-clockwise in the xy plane and that the frames coincide at $t = 0$. The inverse transformation is

$$x = \cos(\Omega t) x' - \sin(\Omega t) y' \quad (34)$$

$$y = \sin(\Omega t) x' + \cos(\Omega t) y' \quad (35)$$

$$z = z' \quad (36)$$

The tensor transformation is

$$I^{ij} = \frac{\partial x^i}{\partial x'^i} \frac{\partial x^j}{\partial x'^j} I'^{i'j'} = \frac{\partial x^i}{\partial x'} \frac{\partial x^j}{\partial x'} I'^{x'x'} \quad (37)$$

where the two summations reduce to a single term. Plug (34) and (35) into (37), we find

$$I^{xx} = \left(\frac{\partial x}{\partial x'} \right)^2 I'^{x'x'} = \frac{1}{12} ML^2 \cos^2(\Omega t) = \frac{1}{12} ML^2 \frac{1}{2} (1 + \cos(2\Omega t)) \quad (38)$$

$$I^{xy} = \frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} I'^{x'x'} = \frac{1}{12} ML^2 \sin(\Omega t) \cos(\Omega t) = \frac{1}{12} ML^2 \frac{1}{2} \sin(2\Omega t) \quad (39)$$

$$I^{yy} = \left(\frac{\partial y}{\partial x'} \right)^2 I'^{x'x'} = \frac{1}{12} ML^2 \sin^2(\Omega t) = \frac{1}{12} ML^2 \frac{1}{2} (1 - \cos(2\Omega t)) \quad (40)$$

$$I^{xz} = I^{yz} = I^{zz} = 0 \quad (41)$$

The metric perturbation of the gravitational wave is

$$\bar{h}^{ij} = \frac{2}{r} \frac{d^2 I^{ij}}{dt^2} = \frac{ML^2 \Omega^2}{3r} \begin{pmatrix} -\cos(2\Omega(t-r)) & -\sin(2\Omega(t-r)) & 0 \\ -\sin(2\Omega(t-r)) & \cos(2\Omega(t-r)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (42)$$

Another way to calculate the quadrupole moment is to treat the rod as composed of infinitesimal point masses $dM = \frac{M}{L} dr$, where $0 < r < \frac{L}{2}$. Then we can use the result for binary stars, in the inertial frame. We found that for one binary

$$dI^{xx} = 2(dM)r^2 \cos^2(\Omega t) = 2\frac{M}{L} dr r^2 \cos^2(\Omega t) \quad (43)$$

integrate binaries of radii $0 < r < \frac{L}{2}$

$$I^{xx} = \int_0^{\frac{L}{2}} 2\frac{M}{L} dr r^2 \cos^2(\Omega t) = \frac{1}{12} ML^2 \cos^2(\Omega t) \quad (44)$$

and the same for I^{xy} and I^{yy} .

3 Energy Loss Of A Binary Star Due To Gravitational Radiation

3.1 Power Emitted

See recitation 11 section 3.

The Power emitted is

$$P = -\frac{2}{5} \left(\frac{M}{R} \right)^5 \quad (45)$$

3.2 Radial Velocity

Newton second law

$$\frac{M^2}{(2R)^2} = M \frac{v^2}{R} \quad (46)$$

The energy of the binary is

$$E = 2 \left(\frac{1}{2} M v^2 \right) - \frac{M^2}{2R} = \frac{M^2}{4R} - \frac{M^2}{2R} = -\frac{M^2}{4R} \quad (47)$$

The power is

$$P = \frac{dE}{dt} = \frac{M^2}{4R^2} \frac{dR}{dt} \quad (48)$$

Compare to (45), yields the radial velocity,

$$\frac{dR}{dt} = -\frac{8}{5} \left(\frac{M}{R} \right)^3 \quad (49)$$

3.3 Period Decrease

From (46) the orbital frequency is

$$\Omega = \left(\frac{M}{4R^3} \right)^{\frac{1}{2}} \quad (50)$$

The period is

$$T = \frac{2\pi}{\Omega} = \frac{4\pi R^{\frac{3}{2}}}{M^{\frac{1}{2}}} \quad (51)$$

Differentiate (51)

$$\frac{dT}{dt} = \frac{4\pi}{M^{\frac{1}{2}}} \frac{3}{2} R^{\frac{1}{2}} \frac{dR}{dt} \quad (52)$$

Plug in (49)

$$\frac{dT}{dt} = -\frac{4\pi}{M^{\frac{1}{2}}} \frac{3}{2} R^{\frac{1}{2}} \frac{8}{5} \left(\frac{M}{R} \right)^3 = -4\pi \frac{3}{2} \frac{8}{5} \frac{M^{\frac{5}{2}}}{R^{\frac{5}{2}}} \quad (53)$$

From (51), substitute $R^{-\frac{5}{2}} = \left(\frac{TM^{\frac{1}{2}}}{4\pi} \right)^{-\frac{5}{3}}$

$$\frac{dT}{dt} = -4\pi \frac{3}{2} \frac{8}{5} M^{\frac{5}{2}} \left(T^{-\frac{5}{3}} \right) \left(\frac{M^{\frac{1}{2}}}{4\pi} \right)^{-\frac{5}{3}} = -\pi \frac{48}{5} \left(\frac{4\pi M}{T} \right)^{\frac{5}{3}} \quad (54)$$

$$\frac{dT}{dt} = -\frac{48}{5} \pi \left(\frac{4\pi M}{T} \right)^{\frac{5}{3}} \quad (55)$$