HW 13

1 Harmonic Oscillator Coefficients

The simple harmonic oscillator of a mass on a spring with constant k follows

$$\ddot{x} + \omega^2 x = 0,$$

where $\omega^2 = k/m$, which is solved by the general solution

$$x(t) = A\cos\omega t + B\sin\omega t = C\cos(\omega t + \phi).$$

Find the parameters A, B, C, ϕ as a function of the initial position and velocity $x(t=0) = x_0$, $v(t=0) = v_0$.

Solution:

Plugging in the initial conditions to the solution we find

$$x_0 = A = C\cos\phi,$$
$$v_0 = \omega B = -\omega C\sin\phi,$$

which leads to

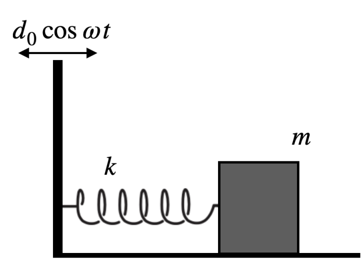
$$A = x_0 \qquad B = \frac{v_0}{\omega},$$

and

$$\tan\phi = -\frac{v_0}{x_0\omega} \qquad C = \frac{x_0}{\cos\phi} = \sqrt{x_0^2 + \frac{v_0 x_0}{\omega}}.$$

2 Driven Damped Oscillator

A mass m is connected to a spring with constant k, which other side's is connected to a wall (see figure). Given that the mass experiences a drag force with constant γ and that the length of the spring at rest is X_0,

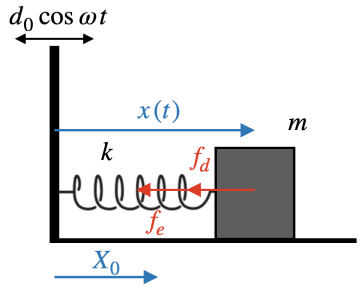


1. Write down all the forces on the mass, assuming position $x(t)$ to the right of the wall.
2. Now, the wall is moving according to the position function $d(t) = d_0\cos\omega t$. Find the position of the mass as a function of time. (you may assume $d_0 \ll X_0$).
3. Draw a graph of the force exerted on the mass as a function of time for the case in which $\omega = \sqrt{k/m}$.

4. Repeat (3) for the case in which $\omega = 2\sqrt{k/m}$. Explain what has changed.

Solution:

1. We assume no friction, thus the forces on the vertical direction are the normal and gravity which cancel one another. On the x direction we have



$$\sum F_x = -k\left[x(t) - X_0\right] - \beta \dot{x}.$$

2. Writing down the equation of motion reads (without the moving wall)

$$m\ddot{x} = -k\left[x(t) - X_0\right] - \beta \dot{x},$$

but now, $X_0 \to X_0 + d_0 \cos \omega t$, thus

$$\ddot{x} + \beta \dot{x} + \frac{k}{m}\left(x - X_0 - d_0 \cos \omega t\right) = 0 \qquad \to \qquad \ddot{x} + 2\gamma \dot{x} + \omega_0^2 \left(x - X_0\right) = \frac{kd_0}{m} \cos \omega t,$$

where we defined $\gamma = \beta/2m$ and $\omega_0^2 = k/m$. For simplicity let us shift the position $x \to x - X_0$, so that

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{kd_0}{m} \cos \omega t.$$

This is the equation of a driven oscillator with intrinsic frequency ω_0 and driving force kd_0, the solution to such equation is composed of the homogeneous and private solutions, where the former is the solution to the equation without the driving force and the latter is a solution which counters the driving force term,

$$x(t) = x_{\text{h}}(t) + x_{\text{p}}(t).$$

The homogeneous solution is that of a damped harmonic oscillator,

$$x_{\text{h}}(t) = \begin{cases} Ae^{-\gamma t}\cos\left(\varpi t + \varphi\right) & \omega_0^2 > \gamma^2 \\ e^{-\gamma t}\left(Ae^{\varpi t} + Be^{-\varpi t}\right) & \omega_0^2 < \gamma^2 \end{cases} \qquad \text{where} \qquad \varpi = \sqrt{\left|\omega_0^2 - \gamma^2\right|}.$$

Whereas the private solution is, in this case, probably a harmonic function similar to the driving force,

$$\begin{aligned} x_{\text{p}}(t) &= C\cos \omega t + D \sin \omega t, \\ \dot{x}_{\text{p}}(t) &= -C\omega \sin \omega t + D\omega \cos \omega t, \\ \ddot{x}_{\text{p}}(t) &= -\omega^2 x, \end{aligned}$$

thus

$$-\omega^2\left(C\cos \omega t + D\sin \omega t\right) + 2\gamma\left(-C\omega \sin \omega t + D\omega \cos \omega t\right) + \omega_0^2\left(C\cos \omega t + D \sin \omega t\right) = \frac{kd_0}{m}\cos \omega t,$$

$$\cos \omega t\left(-C\omega^2 + 2\gamma\omega D + \omega_0^2 C - \frac{kd_0}{m}\right) + \sin \omega t\left(-D\omega^2 - 2\gamma\omega C + \omega_0^2 D\right) = 0.$$

The equation for the second term yields

$$D=\frac{2\gamma\omega}{\omega_0^2-\omega^2}C,$$

plugging into the equation for the first term, recalling that $\omega_0^2=k/m$, gives us

$$C=\frac{\omega_0^2 d_0}{\omega_0^2-\omega^2+2\gamma\omega\frac{2\gamma\omega}{\omega_0^2-\omega^2}}=\frac{\omega_0^2 d_0\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\left(2\gamma\omega\right)^2},$$

$$D=\frac{2\gamma\omega\omega_0^2 d_0}{\left(\omega_0^2-\omega^2\right)^2+\left(2\gamma\omega\right)^2},$$

therefore

$$x_{\mathrm{p}}\left(t\right)=\frac{\omega_0^2 d_0\left[\left(\omega_0^2-\omega^2\right)\cos\omega t+2\gamma\omega\sin\omega t\right]}{\left(\omega_0^2-\omega^2\right)^2+\left(2\gamma\omega\right)^2}.$$

Noting that the homogeneous solution decays with time, after long time we are left with the private solution

$$x\left(\gamma t\gg 1\right)=x_{\mathrm{p}}\left(t\right)=\frac{\omega_0^2 d_0\left[\left(\omega_0^2-\omega^2\right)\cos\omega t+2\gamma\omega\sin\omega t\right]}{\left(\omega_0^2-\omega^2\right)^2+\left(2\gamma\omega\right)^2}+X_0,$$

where we restored the X_0 shift.

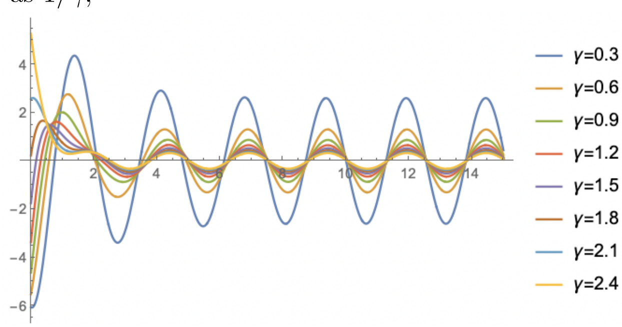
3. The total force follows $F=m\ddot{x}$, thus let us write down the second derivative of the complete solution

$$\ddot{x}=e^{-\gamma t}\left[\gamma^2\cos\varpi t+2\gamma\varpi\sin\varpi t-\varpi^2\cos\varpi t\right]-\frac{\omega^2\omega_0^2 d_0\left[\left(\omega_0^2-\omega^2\right)\cos\omega t+2\gamma\omega\sin\omega t\right]}{\left(\omega_0^2-\omega^2\right)^2+\left(2\gamma\omega\right)^2},$$

where we assumed underdamped regime and initial conditions that fit $A=1$ and $\varphi=0$. For the case $\omega=\sqrt{k/m}=\omega_0$ we find

$$\ddot{x}=e^{-\gamma t}\left[\gamma^2\cos\varpi t+2\gamma\varpi\sin\varpi t-\varpi^2\cos\varpi t\right]-\frac{\omega^3 d_0\sin\omega t}{2\gamma\omega},$$

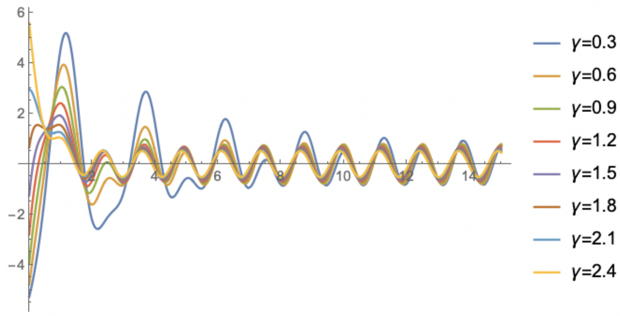
which means that after the initial underdamping transition, we will have a sine function at frequency ω and amplitude that goes as $1/\gamma$,



4. For the case $\omega=2\sqrt{k/m}=2\omega_0$ we find

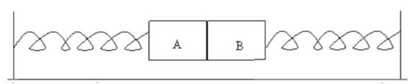
$$\ddot{x}=e^{-\gamma t}\left[\gamma^2\cos\varpi t+2\gamma\varpi\sin\varpi t-\varpi^2\cos\varpi t\right]-\frac{4d_0\left[-\omega_0^2\cos\left(2\omega_0 t\right)+4\gamma\omega_0\sin\left(2\omega_0 t\right)\right]}{1+\left(4\gamma\right)^2},$$

which means that the function will be some combination of cosine and sine, note that the the cosine's amplitude goes as $1/\gamma^2$ whereas the sin's amplitude is proportional to $1/\gamma$.



3 Two masses and two springs

Two identical masses A and B of $M_A = M_B = 1\,kg$ are attached to two identical springs with spring coefficient of $k = 500\,N/m$ as shown in the figure:



In the beginning, both masses are at rest.
At $t = 0$, mass A is displaced an $x = 10\,cm$ to the left and released.
After the release both mass are plastically colliding and start to move together as one body.
Note: for a plastic collision Kinetic Energy is NOT conserved, but Linear Momentum is.

1. How long it takes from when mass A was released and until both mass collides?
2. At what velocity mass A hit mass B?
3. After the collision, when the masses will first stop?
4. Find the time when the connecting point between the two masses will reach $1\,cm$ to the right from their equilibrium point?
5. What is the maximal distance from the equilibrium point that the connecting point between the masses will reach?

Solution:

1. Force equation for mass A at $t = 0$ wil be:

$$M_A \ddot{x}_A = -kx_A$$

with the initial conditions of $x_A(t=0) = -0.1\,m$ and $\dot{x}_A(t=0) = 0\,\frac{m}{sec}$. Therefore

$$x_A(t) = C\cos(\omega_A t + \pi)$$

with $C = 0.1\,m$ and $\omega_A^2 = \frac{k}{M_A}$. Check this solution! (Guess $x_A(t) = C\cos(\omega_A t + \varphi)$ and find C and φ with the initial conditions).
The collision is when $x_A(t) = 0$.

$$0 = C\cos(\omega_A t + \pi)$$

$$\omega_A t + \pi = \frac{3}{2}\pi$$

$$t = \frac{\pi}{2\omega_A} = \frac{\pi}{2}\left(\frac{k}{m}\right)^{-\frac{1}{2}} = \frac{\pi}{2}\left(\frac{500}{1}\right)^{-\frac{1}{2}} \backsimeq 0.07\,sec.$$

2. The velocity of mass A at the time of the collision is

$$\dot{x}_A\left(t=\frac{\pi}{2\omega_A}\right)=-\omega_A C\sin\left(\frac{3}{2}\pi\right)=\omega_A C=0.1\sqrt{500}\backsimeq 2.23\,\frac{m}{sec}.$$

3. Using Linear Momentum Conseravtion

$$M_A\dot{x}_A\left(t=0.07\right)=\left(M_A+M_B\right)V$$

where V is the velocitiy of the body that consist from the two masses after the collision.

$$V=\frac{1}{2}\cdot 2.23\,\frac{m}{sec}\backsimeq 1.11\,\frac{m}{sec}.$$

The Force equation now

$$\left(M_A+M_B\right)\ddot{X}=-kX+k(-X)$$
$$\ddot{X}+\Omega^2X=0$$

where

$$\Omega^2=\frac{2k}{M_A+M_B}=\frac{k}{M_A}=\omega_A^2.$$

Using the initial conditions, $X(t=0.07)=0$ and $\dot{X}(t=0.07)=1.11\,\frac{m}{sec}$,we get

$$X(t)=D\sin\left(\Omega(t-0.07)\right)$$

with $D=0.05\,m$. Now we got everything we needed to find when the masses will first stop:

$$\dot{X}(t)=D\Omega\cos\left(\Omega(t-0.07)\right)$$

where $D\Omega=1.11\,\frac{m}{sec}$. The masses stops when $\dot{X}(t)=0$ which first happens when

$$\Omega(t-0.07)=\frac{\pi}{2}$$

$$t=\frac{\frac{\pi}{2}+0.07\Omega}{\Omega}=\frac{\frac{\pi}{2}+0.07\sqrt{500}}{\sqrt{500}}\backsimeq 0.14\,sec.$$

4. Now we asks when $X(t)=0.01\,m\Rightarrow$

$$D\sin\left(\Omega(t-0.07)\right)=0.01\,m$$
$$\sin\left(\Omega(t-0.07)\right)=\frac{1}{5}$$
$$\Omega(t-0.07)=0.2$$
$$t=0.08\,sec.$$

5. We will recognize that we are actually being asked for the amplitude of X which is

$$D=0.05\,m.$$

4 Oscillating Bowling Ball

A bowling ball weighing $m = 3\,kg$ lies on a smooth floor and is attached to the spring as shown in the figure. The spring coefficient is $k = 111\,N/m$.
An identity bowling ball with a velocity of $v_0 = 10\,m/sec$ collides with it elastically (the collision time is negligible).

1. What is the amplitude of the motion A?
2. What is the period of one cycle?
3. Find $x(t)$ - the location of the attached ball after the collision.
   i.e. $t = 0$ is the collision time.
4. How long will it takes for the attached ball to reach a distance of $\frac{A}{2}$ from equilibrium for the first time?

Solution:

The force equation on the attached ball is

$$m\ddot{x} = -kx$$

with the solution of

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \varphi\right)$$

with $x(t=0) = 0 \Rightarrow \varphi = 0$

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right)$$

and $\dot{x}(t=0)$ we need to find.
We can use energy and momentum conservation during the collision.
Before the collision

$$E = \frac{1}{2}mv_0^2$$

$$p = -v_0$$

After the collision

$$E = \frac{1}{2}m\left(\dot{x}(t=0)^2 + u^2\right)$$

where u is the velocity of the non-attached ball the moment after the collision.

$$p = u - \dot{x}(t=0)$$

From conservation

$$\dot{x}(t=0)^2 + u^2 = v_0^2$$

$$u - \dot{x}(t=0) = -v_0$$

using algebra we get

$$\dot{x}(t=0) = v_0$$

1. Finding the amplitude A using the initial condition $\dot{x}(t=0) = v_0$

$$A\sqrt{\frac{k}{m}} = v_0$$

$$A = v_0\sqrt{\frac{m}{k}} = 10\sqrt{\frac{3}{111}} = 1.64\,m$$

2. The period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \cong 1\,sec$$

3. We already found $x(t)$ and now we also know what is A

$$x(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

4. We need to find out when $|x(t)| = A/2$

$$\left|\frac{v_0}{2}\sqrt{\frac{m}{k}}\right| = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

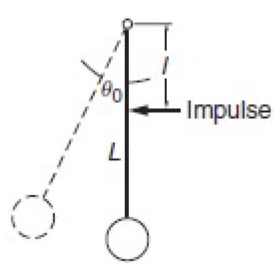
$$\sin\left(\sqrt{\frac{k}{m}}t\right) = \frac{1}{2}$$

$$\sqrt{\frac{k}{m}}t = \frac{\pi}{6}$$

$$t = \frac{\pi}{6}\sqrt{\frac{m}{k}} = 0.083\,sec$$

5 Grandfather clock

The pendulum of a grandfather clock activates an escapement mechanism every time it passes through the vertical.
The escapement is under tension (provided by a hanging weight) and gives the pendulum a small impulse a distance l from the pivot.
The energy transferred by this impulse compensates for the energy dissipated by friction, so that the pendulum swings with a constant amplitude.



1. What is the impulse needed to sustain the motion of a pendulum of length L and mass m, with an amplitude of swing θ_0 and quality factor Q?

$$Q = \frac{\text{Energy of the oscillator}}{\text{Energy dissipate per radian}}$$

2. Why is it desirable for the pendulum to engage the escapement as it passes vertical rather than at some other point of the cycle?

Solution:

The pendulum loses mechanical energy ΔE as it swings, due to friction.
Its speed decreases slightly from v_0 to some v_1 during a half cycle.
The escapement provides an impulse every period to make up for the loss.

1. The energy loss from the moment the pendulum swings to the left with tangential velocity of v_0 until the moment the pendulum returns to equilibrium with velocity of v_1 after 1 half of a cycle (π radians).

$$\Delta E = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2$$

   Assuming low rate of dissipation and therefore $v_0 \approx v_1$. We can treat the energy of the pendulum to be almost constant during half cycle $E \sim const = \frac{1}{2}mv_0^2$

$$Q = \frac{\frac{1}{2}mv_0^2}{\Delta E/\pi} = \frac{\frac{1}{2}m\pi v_0^2}{\frac{1}{2}m\left(v_0^2 - v_1^2\right)} = \frac{\pi v_0^2}{\left(v_0 - v_1\right)\left(v_0 + v_1\right)} \approx \frac{\pi v_0^2}{2v_0\left(v_0 - v_1\right)}$$

$$\Delta v = v_0 - v_1 = \frac{\pi v_0}{2Q}$$

   the required impulse $I = \Delta p = m\Delta v = \frac{m\pi v_0}{2Q}$.
   The pendulum motion is

$$\theta = \theta_0 \sin \omega t$$

   where $\omega = \sqrt{g/L}$. The speed v_0 at the beginning of the upswing is

$$v_0 = L\dot{\theta} = L\left(\omega\theta_0\right) = \sqrt{gL}\theta_0$$

$$I = m\frac{\pi\theta_0}{2Q}\sqrt{gL}$$

2. The impulse I produces a change in speed $\Delta v = I/m$, so that the energy is increased by ΔE_I . where

$$\Delta E_I = \frac{1}{2}m\left(v + \Delta v\right)^2 - \frac{1}{2}mv^2 = mv\Delta v + \frac{1}{2}m\left(\Delta v\right)^2 = Iv + \frac{I^2}{2m}$$

   The point in the cycle where the impulse acts can vary due to mechanical imperfections. To minimize this effect, the impulse should be applied when v is not changing to first order with respect to θ, which is at the bottom of the swing.