Homework 2 - Introduction to Quantum Physics

Note:

It is often useful to remember the value $hc = 1.24 \times 10^{-6}\text{eV} \cdot \text{m}$.

De Broglie waves

Question 1:

A thermal neutron has a speed v at temperature $T = 300\text{K}$ and kinetic energy $\frac{1}{2}m_n v^2 = \frac{3}{2}k_B T$, where $k_B = 8.6 \times 10^{-5}\text{eV} \cdot \text{K}^{-1}$ is Boltzmann constant. Calculate its de Broglie wavelength. State whether a beam of these neutrons could be diffracted by a crystal, and why? (Neutron mass is $m_n = 940\text{MeV}/c^2$)

Solution:

Replacing $p = m_n v$ we find

$$\frac{p^2}{m_n} = 3k_B T \quad \Rightarrow \quad pc = \sqrt{3k_B T m_n c^2} = 8542\text{eV},$$

then

$$\lambda = \frac{hc}{pc} = 1.45\text{Å}.$$

Question 2:

Show that the de Broglie wave length for neutrons is given by $\lambda = \frac{0.286\text{Å}}{\sqrt{T/\text{eV}}}$, where T is its kinetic energy.

Solution:

The momentum of a neutron can be expressed in terms of its kinetic energy

$$T = \frac{p^2}{2m} \quad \Rightarrow \quad p = \sqrt{2mT}.$$

Thus, its de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{2mc^2}}T^{-1/2} = \frac{1.24 \times 10^4\text{eV} \cdot \text{Å}}{\sqrt{2 \times 940 \times 10^6\text{eV}}}T^{-1/2} = \frac{0.286\text{Å}}{\sqrt{T/\text{eV}}}.$$

Question 3:

The relation for total energy E and momentum p for a relativistic particle is $E^2 = p^2c^2 + m_0^2c^4$, where m_0 is the rest mass and c is the speed of light. Using the relativistic relations $E = \hbar\omega$ and $p = \hbar k$, where ω is the angular frequency, k is the wave number and $\hbar = h/2\pi$, show that the product of group velocity v_g and the phase velocity v_p is equal to c^2, that is $v_g v_p = c^2$.

Solution:

Recalling that

$$v_p = \frac{\omega}{k} \quad \text{and} \quad v_p = \frac{d\omega}{dk},$$

and using the relativistic relation

$$E^2 = \hbar^2\omega^2 = \hbar^2 k^2 c^2 + m_0^2 c^4 \quad \Rightarrow \quad \omega = \sqrt{k^2c^2 + m_0^2 c^4/\hbar^2},$$

we find

$$v_p = \frac{\omega}{k} \quad \text{and} \quad v_g = \frac{kc^2}{\omega} \quad \Rightarrow \quad v_g v_p = \frac{kc^2}{\omega}\frac{\omega}{k} = c^2.$$

Double Slit Experiment

Question 4:

Consider the double slits experiment with photons, only this time the wavelength of the photons is gradually increased. What would be the wavelength for which there wont be visible interference?

Solution:

In class we've seen that the intensity at angle θ on the screen can be expressed as

$$I \sim \cos^2\left(\frac{kd\sin\theta}{2}\right),$$

which vanishes when

$$\frac{kd\sin\theta}{2} = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm1, \pm2\ldots$$

This equation describes the destructive interference lines, for which the 0th order is

$$kd\sin\theta = \pi.$$

In order for the 0th order to be outside the screen (i.e. all the other interference orders will be out as well) we require $\theta_c = \pi/2$, which leads to

$$\sin\theta = \frac{\lambda}{2d}$$

$$k_c = \frac{2\pi}{\lambda} = \frac{\pi}{d} \quad \rightarrow \quad \lambda_c = 2d,$$

thus, if $\lambda < 2d$ we will see the interference patter, while for $\lambda \geq 2d$ we will not see it.

Structure of The Atom

Question 5:

1. By using the de Broglie relation, derive the Bohr condition $mvr = n\hbar$ for the angular momentum of an electron in a hydrogen atom.
2. Use this expression to show that the allowed electron energy states in hydrogen atom can be written $E_n = -\frac{me^4}{8\varepsilon_0^2 h^2 n^2}$.
3. How would this expression be modified for the case of a triply ionized beryllium atom Be(Z = 4)?
4. Calculate the ionization energy in eV of Be^{+3} (ionization energy of hydrogen = 13.6 eV)

Solution:

1. Stationary orbits will be such that the circumference of a circular orbit is equal to an integral number of de Broglie wavelength so that constructive interference may take place i.e. $2\pi r = n\lambda$. But $\lambda = h/p$, then

$$L = rp = \frac{hn}{2\pi} = n\hbar.$$

2. See derivation in class notes.
3. The only change will be in the Coulomb force which will read

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2},$$

thus the energy levels will be

$$E_n = -\frac{mZ^2e^4}{8\varepsilon_0^2 h^2 n^2} \quad \stackrel{Z=3}{\Rightarrow} \quad E_n = -\frac{9me^4}{8\varepsilon_0^2 h^2 n^2}.$$

4. The ionization energy is just Z^2 times the hydrogen energy

$$E_I = 9 \times 13.6\text{eV} = 122.4\text{eV}.$$