

Homework 4 - Wave function

Note: The Schrödinger equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi,$$

while the *time-independent Schrödinger* equation is found by taking $\Psi(x, t) = \psi(x) e^{-i\frac{E}{\hbar}t}$,

$$E\psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi.$$

Question 1:

Consider a free particle moving in one dimension, i.e. $\omega = \hbar k^2/2m$. At time $t = 0$ its wave function is

$$\psi(x) = N \left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2} \right],$$

where α and x_0 are real parameters such that $\{\alpha, x_0\} > 0$.

1. Compute the normalization factor $|N|$ and the momentum wave function $\tilde{\psi}(k)$.
2. Find the time dependent wave function $\Psi(x, t)$ at any time $t > 0$. *Hint:* Use the notation $z = 1 + i\left(\frac{\hbar\alpha t}{m}\right)$.
3. Write down the position probability density and discuss the physical interpretation of each term.
4. Obtain an expression for the probability current, which is defined as

$$J(x, t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right).$$

Solution:

1. Using the normalization condition

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1,$$

gives us

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi|^2 dx &= |N|^2 \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2} \right]^2 dx \\ &= |N|^2 \int_{-\infty}^{\infty} \left[e^{-\alpha(x+x_0)^2} + 2e^{-\alpha(x^2+x_0^2)} + e^{-\alpha(x-x_0)^2} \right] dx \\ &= |N|^2 \sqrt{\frac{\pi}{\alpha}} \left[1 + 2e^{-\alpha x_0^2} + 1 \right], \end{aligned}$$

thus

$$\boxed{|N| = \frac{1}{\sqrt{2}} \left(\frac{\alpha}{\pi} \right)^{1/4} (1 + \exp[-\alpha x_0^2])^{-1/2}}.$$

Whereas the Fourier transform is

$$\begin{aligned} \tilde{\psi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \\ &= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2} \right] e^{-ikx} dx \end{aligned}$$

completing the square in the powers reads

$$\begin{aligned}
-\frac{\alpha}{2}(x \pm x_0)^2 - ikx &= -\frac{\alpha}{2}(x^2 + 2(ik/\alpha \pm x_0)x + x_0^2) \\
&= -\frac{\alpha}{2}\left(x^2 + 2(ik/\alpha \pm x_0)x + (ik/\alpha \pm x_0)^2\right) + \frac{\alpha}{2}(ik/\alpha \pm x_0)^2 - \frac{\alpha}{2}x_0^2 \\
&= -\frac{\alpha}{2}(x + ik/\alpha \pm x_0)^2 - \frac{1}{2\alpha}(k^2 \mp 2i\alpha x_0 k - \alpha^2 x_0^2) - \frac{\alpha}{2}x_0^2 \\
&= -\frac{\alpha}{2}(x + ik/\alpha \pm x_0)^2 - \frac{k^2}{2\alpha} \pm ix_0 k
\end{aligned}$$

thus

$$\begin{aligned}
\tilde{\psi}(k) &= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha}} \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+ik/\alpha+x_0)^2} e^{ix_0 k} + e^{-\frac{\alpha}{2}(x+ik/\alpha-x_0)^2} e^{-ix_0 k} \right] dx \\
&= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha}} \sqrt{\frac{2\pi}{\alpha}} [e^{ix_0 k} + e^{-ix_0 k}] \\
&= \frac{2N}{\sqrt{\alpha}} e^{-\frac{k^2}{2\alpha}} \cos x_0 k,
\end{aligned}$$

plugging in the normalization factor we get

$$\boxed{\tilde{\psi}(k) = \frac{\sqrt{2}}{(\pi\alpha)^{1/4}} \frac{e^{-\frac{k^2}{2\alpha}}}{\sqrt{1 + \exp[-\alpha x_0^2]}} \cos x_0 k.}$$

2. Taking the inverse Fourier transform, considering the time evolution of a free particle, i.e. $\omega = \hbar k^2/2m$, we have

$$\begin{aligned}
\Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{i(kx - \omega t)} dk \\
&= \frac{2N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\alpha}} \cos x_0 k e^{i(kx - \frac{\hbar t}{2m} k^2)} dk \\
&= \frac{N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\alpha}(1 + i\frac{\hbar t\alpha}{m}) + ikx} (e^{ikx_0} + e^{-ikx_0}) dk \\
&= \frac{N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} \left(e^{-\frac{z}{2\alpha}(k^2 - 2i\frac{\alpha k}{z}(x+x_0))} + e^{-\frac{z}{2\alpha}(k^2 - 2i\frac{\alpha k}{z}(x-x_0))} \right) dk,
\end{aligned}$$

where $z(t) \equiv 1 + i\frac{\hbar\alpha t}{m}$. Looking at the powers

$$\begin{aligned}
-\frac{z}{2\alpha} \left(k^2 - 2i\frac{\alpha k}{z}(x \pm x_0) \right) &= -\frac{z}{2\alpha} \left(k^2 - 2i\frac{\alpha}{z}(x \pm x_0)k + \frac{(i\alpha)^2}{z^2}(x \pm x_0)^2 \right) - \frac{\alpha}{2z}(x \pm x_0)^2 \\
&= -\frac{z}{2\alpha} \left(k^2 - i\frac{\alpha}{z}(x \pm x_0) \right)^2 - \frac{\alpha}{2z}(x \pm x_0)^2,
\end{aligned}$$

thus

$$\begin{aligned}
\Psi(x, t) &= \frac{N}{\sqrt{2\pi\alpha}} \sqrt{\frac{2\pi\alpha}{z}} \left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2} \right) \\
&\boxed{\Psi(x, t) = \frac{N}{\sqrt{z(t)}} \left(e^{-\frac{\alpha}{2z(t)}(x+x_0)^2} + e^{-\frac{\alpha}{2z(t)}(x-x_0)^2} \right),}
\end{aligned}$$

which are the same initial Gaussians we started with, evolved in time.

3. The probability density is

$$\left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2} \right) \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) =$$

$$\begin{aligned}
\rho(x,t) &= \Psi^* \Psi \\
&= \frac{N^2}{\sqrt{|z|^2}} \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) \\
&= \frac{N^2}{\sqrt{|z|^2}} \left(e^{-\frac{\alpha \operatorname{Re}[z]}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha \operatorname{Re}[z]}{|z|^2}(x-x_0)^2} \right. \\
&\quad \left. + e^{-\frac{\alpha}{|z|^2}(\operatorname{Re}[z]x^2 + \operatorname{Re}[z]x_0^2 + i\operatorname{Im}[z]2x_0x)} + e^{-\frac{\alpha}{|z|^2}(\operatorname{Re}[z]x^2 + \operatorname{Re}[z]x_0^2 - i\operatorname{Im}[z]2x_0x)} \right),
\end{aligned}$$

Recalling that $\operatorname{Re}[z] = 1$ and $\operatorname{Im}[z] = \hbar\alpha t/m$ we have

$$\rho(x,t) = \frac{N^2}{\sqrt{|z|^2}} \left[e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)} \left(e^{\frac{i2\hbar t x_0 \alpha^2}{m|z|^2}x} + e^{-\frac{i2\hbar t x_0 \alpha^2}{m|z|^2}x} \right) \right]$$

$$\rho(x,t) = \frac{N^2}{\sqrt{|z|^2}} \left[\underbrace{e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x-x_0)^2}}_{\text{original Gaussians}} + 2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)} \underbrace{\cos\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right)}_{\text{Interference term}} \right].$$

4. Note that

$$J(x,t) = \frac{i\hbar}{2m} \left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right) = -\frac{\hbar}{m} \operatorname{Im} \left[\Psi \frac{\partial \Psi^*}{\partial x} \right].$$

Taking the spatial derivative of Ψ^* ,

$$\begin{aligned}
\Psi \frac{\partial \Psi^*}{\partial x} &= -\frac{\alpha}{z^*} \frac{N^2}{\sqrt{|z|^2}} \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) \left((x+x_0) e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + (x-x_0) e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) \\
&= -\frac{\alpha}{z^*} \frac{N^2}{\sqrt{|z|^2}} \left[(x+x_0) e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0) e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} \right. \\
&\quad \left. + 2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)} \left(x \cos\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) - ix_0 \sin\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) \right) \right], \\
&= -\frac{(1+i\frac{\hbar\alpha t}{m})\alpha N^2}{|z|^3} \left[(x+x_0) e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0) e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} \right. \\
&\quad \left. + 2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)} \left(x \cos\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) - ix_0 \sin\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) \right) \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
J(x,t) &= \frac{\alpha N^2}{|z|^3} \frac{\hbar}{m} \left[\frac{\hbar\alpha t}{m} \left((x+x_0) e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0) e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} \right) \right. \\
&\quad \left. e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)} \left(\frac{\hbar\alpha t}{m} x \cos\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) - 2x_0 \sin\left(2\frac{\hbar t}{m} \frac{\alpha^2 x_0 x}{|z|^2}\right) \right) \right].
\end{aligned}$$

Question 2:

Consider a free particle, i.e. $\omega = \hbar k^2/2m$, that is initially (at $t = 0$) extremely well localized at the origin (i.e. at $x = 0$) and has a Gaussian wave function,

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}} e^{ik_0 x},$$

where α is very large ($\alpha \gg 1$), and k_0 is a real constant.

1. Write down position probability density $\rho(x, 0)$ and show that

$$\lim_{\alpha \rightarrow \infty} \rho(x, 0) = \delta(x).$$

2. Keep α finite and calculate $\tilde{\Psi}(x, t)$ at times $t > 0$ and the k -space probability density $\tilde{\rho}(k, t)$. What is the most momentum of the particle?

3. Calculate $\Psi(x, t)$ at times $t > 0$ and the position probability density $\rho(x, t)$. What is the most probable position as a function of time? What happens at the limit $t \rightarrow \infty$?

4. Calculate the expectation values $\langle p(t) \rangle$ and $\langle x(t) \rangle$ and show that they follow the classical laws of motion for a free particle, that is

$$\langle p \rangle_t = m \frac{d\langle x \rangle_t}{dt} \quad \text{and} \quad \frac{d\langle p \rangle_t}{dt} = m \frac{d^2\langle x \rangle_t}{dt^2} = 0.$$

5. Calculate the expectation values $\langle p^2(t) \rangle$ and $\langle x^2(t) \rangle$ and check consistency with the position-momentum uncertainty principle, i.e. $(\Delta x(t))^2 (\Delta p(t))^2 \geq \hbar^2/4$.

6. What is the physical meaning of the quantity $\tau = \frac{\Delta x(t)}{|d\langle x(t) \rangle/dt|}$ (beyond the immediate conclusion from units). Show that it is consistent with the time-energy uncertainty principle $\tau \Delta E \geq \hbar/2$.

Solution:

1. The position probability is

$$\rho(x) = \psi^* \psi = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2}.$$

Taking the integral over a multiplication of a function $f(x)$ and the limit

$$\int_{-\infty}^{\infty} f(x) \lim_{\alpha \rightarrow \infty} \rho(x) dx = \int_{-\infty}^{\infty} \lim_{\alpha \rightarrow \infty} f(x) \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} dx,$$

changing variables to $u = \sqrt{\alpha}x$, thus $dx = du/\sqrt{\alpha}$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \lim_{\alpha \rightarrow \infty} \rho(x) dx &= \int_{-\infty}^{\infty} \lim_{\alpha \rightarrow \infty} f\left(\frac{u}{\sqrt{\alpha}}\right) \frac{1}{\sqrt{\pi}} e^{-u^2} du \\ &= \int_{-\infty}^{\infty} f(0) \frac{1}{\sqrt{\pi}} e^{-u^2} du \\ &= f(0), \end{aligned}$$

which proves that

$$\boxed{\lim_{\alpha \rightarrow \infty} \rho(x, 0) = \delta(x)}.$$

2. The Fourier transform is

$$\begin{aligned} \tilde{\psi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\frac{\alpha x^2}{2}} e^{-i(k-k_0)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2} \left(x + i\frac{(k-k_0)}{\alpha}\right)^2 - \frac{(k-k_0)^2}{2\alpha}} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{(k-k_0)^2}{2\alpha}} \sqrt{\frac{2\pi}{\alpha}}, \end{aligned}$$

thus

$$\boxed{\tilde{\Psi}(k, t) = \frac{e^{-\frac{(k-k_0)^2}{2\alpha}}}{(\alpha\pi)^{1/4}} e^{-i\frac{\hbar k^2}{2m}t}}.$$

The k -space probability density is

$$\tilde{\rho}(k, t) = \tilde{\Psi}(k, t) \Psi(k, t) = \frac{e^{-\frac{(k-k_0)^2}{\alpha}}}{\sqrt{\alpha\pi}},$$

therefore, the most probable momentum is $p_{\max} = \hbar k_{\max} = \hbar k_0$.

3. Taking the inverse Fourier transform we find

$$\begin{aligned} \Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(k, t) e^{ikx} dk \\ &= \frac{1}{(\alpha\pi)^{1/4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\alpha}} e^{-i\frac{\hbar k^2}{2m}t} e^{ikx} dk \\ &= \frac{1}{(\alpha\pi)^{1/4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha} (k^2 - 2(\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m})k) - \frac{1}{2\alpha} k_0^2} dk \\ &= \frac{1}{(\alpha\pi)^{1/4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha} (k - \frac{k_0+i\alpha x}{1+i\hbar t\alpha/m})^2 + \frac{1}{2\alpha} (1+i\hbar t\alpha/m) (\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m})^2 - \frac{1}{2\alpha} k_0^2} dk \\ &= \frac{1}{(\alpha\pi)^{1/4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha} (k - \frac{k_0+i\alpha x}{1+i\hbar t\alpha/m})^2 - \frac{\alpha}{2} \frac{(x - ik_0/\alpha)^2}{1+i\hbar t\alpha/m} - \frac{1}{2\alpha} k_0^2} dk \\ &= \frac{1}{(\alpha\pi)^{1/4}} \sqrt{\frac{\alpha}{1+i\hbar t\alpha/m}} e^{-\frac{\alpha}{2} \frac{(x - ik_0/\alpha)^2}{1+i\hbar t\alpha/m} - \frac{1}{2\alpha} k_0^2} \\ &= \frac{1}{(\alpha\pi)^{1/4}} \sqrt{\frac{\alpha}{1+i\hbar t\alpha/m}} e^{-\frac{\alpha}{2} \frac{x^2 - k_0^2/\alpha^2 - 2k_0 x \hbar t/m - 2ik_0 x/\alpha - (x^2 - k_0^2/\alpha^2) i\hbar t\alpha/m}{1+\hbar^2 t^2 \alpha^2/m^2} - \frac{1}{2\alpha} k_0^2}. \end{aligned}$$

Thus, the time dependent probability density is

$$\rho(x, t) = \Psi^* \Psi = \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1+\hbar^2 t^2 \alpha^2/m^2}} e^{-\alpha \frac{x^2 - k_0^2/\alpha^2 - 2k_0 x \hbar t/m}{1+\hbar^2 t^2 \alpha^2/m^2} - \frac{1}{\alpha} k_0^2},$$

looking at the power we have

$$\begin{aligned} -\alpha \frac{x^2 - k_0^2/\alpha^2 - 2k_0 x \hbar t/m}{1+\hbar^2 t^2 \alpha^2/m^2} - \frac{1}{2\alpha} k_0^2 &= -\alpha \frac{(x - k_0 \hbar t/m)^2 - \frac{k_0^2}{\alpha^2} (1+\hbar^2 t^2 \alpha^2/m^2)}{1+\hbar^2 t^2 \alpha^2/m^2} - \frac{1}{\alpha} k_0^2 \\ &= -\alpha \frac{(x - k_0 \hbar t/m)^2}{1+\hbar^2 t^2 \alpha^2/m^2}, \end{aligned}$$

so that

$$\rho(x, t) = \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1+\hbar^2 t^2 \alpha^2/m^2}} e^{-\alpha \frac{(x - k_0 \hbar t/m)^2}{1+\hbar^2 t^2 \alpha^2/m^2}}.$$

The most probable position is $x_{\max} = k_0 \hbar t/m$. When taking the limit $t \rightarrow \infty$ we find

$$\lim_{t \rightarrow \infty} \rho(x, t) = \sqrt{\frac{1}{\alpha\pi}} \frac{m}{\hbar t} e^{-\frac{k_0^2}{\alpha}},$$

i.e. the distribution does not depend on the position after a long time, it goes to zero at each point as well (since it smears over the entire space).

4. The position expectation value is

$$\begin{aligned} \langle x(t) \rangle &= \int_{-\infty}^{\infty} \Psi^* x \Psi dx \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1+\hbar^2 t^2 \alpha^2/m^2}} \int_{-\infty}^{\infty} x e^{-\alpha \frac{(x - k_0 \hbar t/m)^2}{1+\hbar^2 t^2 \alpha^2/m^2}} dx \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1+\hbar^2 t^2 \alpha^2/m^2}} \int_{-\infty}^{\infty} (u + k_0 \hbar t/m) e^{-\alpha \frac{u^2}{1+\hbar^2 t^2 \alpha^2/m^2}} du, \end{aligned}$$

for which the first term vanishes since we symmetrically integrate an odd function, and we are left with

$$\langle x(t) \rangle = \frac{\hbar k_0}{m} t.$$

While the momentum expectation value is

$$\langle p(t) \rangle = \hbar \int_{-\infty}^{\infty} \tilde{\Psi}^* k \tilde{\Psi} dk = \frac{\hbar}{\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} k e^{-\frac{(k-k_0)^2}{\alpha}} dk = \frac{\hbar}{\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} (s + k_0) e^{-\frac{s^2}{\alpha}} ds,$$

just like in the position calculation we find

$$\langle p(t) \rangle = \hbar k_0.$$

It is straightforward to see that these follow the classical laws of motion,

$$\langle p \rangle_t = m \frac{d \langle x \rangle_t}{dt} \quad \text{and} \quad \frac{d \langle p(t) \rangle}{dt} = m \frac{d^2 \langle x \rangle_t}{dt^2} = 0.$$

5. The second moment for the position distribution is

$$\begin{aligned} \langle x^2(t) \rangle &= \int_{-\infty}^{\infty} \Psi^* x^2 \Psi dx \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1 + \hbar^2 t^2 \alpha^2 / m^2}} \int_{-\infty}^{\infty} x^2 e^{-\alpha \frac{(x - k_0 \hbar t / m)^2}{1 + \hbar^2 t^2 \alpha^2 / m^2}} dx \\ &= \sqrt{\frac{\alpha}{\pi}} \sqrt{\frac{1}{1 + \hbar^2 t^2 \alpha^2 / m^2}} \int_{-\infty}^{\infty} \left(u^2 + 2 \frac{k_0 \hbar t}{m} u + \frac{k_0^2 \hbar^2 t^2}{m^2} \right) e^{-\alpha \frac{u^2}{1 + \hbar^2 t^2 \alpha^2 / m^2}} du, \end{aligned}$$

evaluating the nonzero integrated terms yields

$$\langle x^2(t) \rangle = \left(\frac{\hbar k_0}{m} t \right)^2 + \frac{1 + \hbar^2 t^2 \alpha^2 / m^2}{2\alpha},$$

thus

$$(\Delta x(t))^2 = \langle x^2(t) \rangle - \langle x(t) \rangle^2 = \frac{1 + \hbar^2 t^2 \alpha^2 / m^2}{2\alpha}.$$

While the second moment for the momentum distribution is

$$\langle p^2(t) \rangle = \hbar^2 \int_{-\infty}^{\infty} \tilde{\Psi}^* k^2 \tilde{\Psi} dk = \frac{\hbar^2}{\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} k^2 e^{-\frac{(k-k_0)^2}{\alpha}} dk = \frac{\hbar^2}{\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} (s^2 + 2sk_0 + k_0^2) e^{-\frac{s^2}{\alpha}} ds,$$

evaluating the nonzero integrated terms yields

$$\langle p^2(t) \rangle = \hbar^2 k_0^2 + \frac{\hbar^2 \alpha}{2},$$

thus

$$(\Delta p(t))^2 = \langle p^2(t) \rangle - \langle p(t) \rangle^2 = \frac{\hbar^2 \alpha}{2}.$$

So the position-momentum uncertainty holds,

$$[\Delta x(t) \Delta p(t)]^2 = \frac{\hbar^2}{4} (1 + \hbar^2 t^2 \alpha^2 / m^2) > \frac{\hbar^2}{4}.$$

6. The quantity

$$\tau = \frac{\Delta x(t)}{|d \langle x(t) \rangle / dt|} = \frac{m}{\hbar k_0} \sqrt{\frac{1 + \hbar^2 t^2 \alpha^2 / m^2}{2\alpha}}$$

quantifies the time it takes the expectation value to shift past its spread, i.e. the typical spreading time. It is minimal at $t = 0$ when its value is $\tau(t = 0) = \frac{m}{\hbar k_0 \sqrt{2\alpha}}$.

Question 3:

Consider a system with a real Hamiltonian that occupies a stationary state having a real wave function both at time $t = 0$ and $t = t_1$. Meaning, the wave function satisfies:

$$\Psi^*(x, 0) = \Psi(x, 0) \quad \text{and} \quad \Psi^*(x, t_1) = \Psi(x, t_1).$$

1. Show that such a system must be periodic, i.e. that there exists a period T such that

$$\Psi(x, t) = \Psi(x, t + T) \quad \forall t.$$

2. Calculate the period T .

3. Show that for such a system, the energy eigenvalues must be integer multiples of $2\pi/T$.

Hint: Assume the state $\Psi(x, t)$ has some defined energy E , so that $\Psi(x, t) = e^{-i\frac{E}{\hbar}t}\psi(x)$, then in (3) show it must obey the given condition.

Solution:

1. The time dependent wave function at time t_1 is

$$\Psi(x, t_1) = e^{-i\frac{E}{\hbar}t_1}\psi(x),$$

which, for a real Hamiltonian (since a real Hamiltonian \hat{H} has a real eigenvalue E), has the complex conjugate

$$\Psi^*(x, t_1) = e^{i\frac{E}{\hbar}t_1}\psi^*(x) = e^{i\frac{E}{\hbar}t_1}\psi(x),$$

where we used $\psi(x) = \Psi(x, 0)$. Thus

$$e^{i\frac{E}{\hbar}t_1}\psi(x) = e^{-i\frac{E}{\hbar}t_1}\psi(x) \quad \rightarrow \quad e^{i\frac{E}{\hbar}2t_1} = 1.$$

For a general time t the wave function reads

$$\Psi(x, t) = e^{-i\frac{E}{\hbar}t}\psi(x) = e^{-i\frac{E}{\hbar}t}e^{-i\frac{E}{\hbar}2t_1}\psi(x) = e^{-i\frac{E}{\hbar}(t+2t_1)}\psi(x) = \Psi(x, t + 2t_1),$$

thus

$$\boxed{\Psi(x, t) = \Psi(x, t + T) \quad \text{where} \quad T = 2t_1}.$$

2. As shown in (1) the period is $T = 2t_1$.

3. Using the equation we found for T ,

$$e^{i\frac{E}{\hbar}T} = 1 \quad \rightarrow \quad \frac{E}{\hbar}T = 2\pi n \quad \rightarrow \quad \boxed{E = \frac{2\pi\hbar}{T}n, \quad n \in \mathbb{Z}}.$$

Question 4:

Consider a normalized wave function $\psi_0(x)$. Assume that the system is in a state described by the wave function

$$\psi(x) = C_1\psi_0(x) + C_2\psi_0^*(x),$$

where C_1 and C_2 are two known complex constants.

1. Write down the condition for the normalization of $\psi(x)$ in terms of the known complex integral,

$$D = \int_{-\infty}^{\infty} \psi_0(x)\psi_0(x)dx.$$

2. Obtain an expression for the probability current $J(x)$ for the state $\psi(x)$. Use a polar representation of complex numbers, i.e. $\psi_0(x) = f(x)e^{i\theta(x)}$.

3. Calculate the expectation value of the momentum $\langle p \rangle$, assuming that $\hat{p} = -i\hbar(\partial/\partial x)$, and show that

$$\langle \psi | \hat{p} \psi \rangle = m \int_{-\infty}^{\infty} J(x) dx.$$

Show that both probability current and momentum vanish if $|C_1| = |C_2|$.

Solution:

1. The normalization condition is

$$\begin{aligned} \langle \psi | \psi \rangle &= \langle C_1 \psi_0(x) + C_2 \psi_0^*(x) | C_1 \psi_0(x) + C_2 \psi_0^*(x) \rangle \\ &= |C_1|^2 \langle \psi_0 | \psi_0 \rangle + C_1 C_2^* \langle \psi_0^* | \psi_0 \rangle + C_1^* C_2 \underbrace{\langle \psi_0 | \psi_0^* \rangle}_{\langle \psi_0^* | \psi_0 \rangle^*} + |C_2|^2 \langle \psi_0^* | \psi_0^* \rangle \\ &= |C_1|^2 + C_1 C_2^* D + C_1^* C_2 D^* + |C_2|^2, \end{aligned}$$

thus

$$\boxed{\langle \psi | \psi \rangle = |C_1|^2 + |C_2|^2 + 2\text{Re}[C_1 C_2^* D] = 1.}$$

2. Using the polar representation we have

$$\psi_0(x) = f(x) e^{i\theta(x)} \quad \rightarrow \quad \psi(x) = f(x) \left(C_1 e^{i\theta(x)} + C_2 e^{-i\theta(x)} \right)$$

where $f(x)$ and $\theta(x)$ are a real valued functions. Therefore the differentiation of ψ reads

$$\frac{\partial \psi}{\partial x} = f'(x) \left(C_1 e^{i\theta(x)} + C_2 e^{-i\theta(x)} \right) + i\theta'(x) f(x) \left(C_1 e^{i\theta(x)} - C_2 e^{-i\theta(x)} \right).$$

The probability current is

$$J(x) = \frac{\hbar}{m} \text{Im} \left[\psi^* \frac{\partial \psi}{\partial x} \right],$$

where

$$\begin{aligned} \psi^* \frac{\partial \psi}{\partial x} &= f(x) \left(C_1^* e^{-i\theta(x)} + C_2^* e^{i\theta(x)} \right) \left(f'(x) \left(C_1 e^{i\theta(x)} + C_2 e^{-i\theta(x)} \right) + i\theta'(x) f(x) \left(C_1 e^{i\theta(x)} - C_2 e^{-i\theta(x)} \right) \right) \\ &= f f' \left(C_1^* e^{-i\theta} + C_2^* e^{i\theta} \right) \left(C_1 e^{i\theta} + C_2 e^{-i\theta} \right) + i\theta' f^2 \left(C_1^* e^{-i\theta} + C_2^* e^{i\theta} \right) \left(C_1 e^{i\theta} - C_2 e^{-i\theta} \right) \\ &= f f' \left(|C_1|^2 + C_1 C_2^* e^{2i\theta} + C_1^* C_2 e^{-2i\theta} + |C_2|^2 \right) + i\theta' f^2 \left(|C_1|^2 + C_1 C_2^* e^{2i\theta} - C_1^* C_2 e^{-2i\theta} - |C_2|^2 \right) \\ &= f f' \underbrace{\left(|C_1|^2 + 2\text{Re}[C_1 C_2^* e^{2i\theta}] + |C_2|^2 \right)}_{\text{real}} + i\theta' f^2 \left(|C_1|^2 - |C_2|^2 + 2i\text{Im}[C_1 C_2^* e^{2i\theta}] \right), \end{aligned}$$

taking only the imaginary part of $\psi^* \frac{\partial \psi}{\partial x}$, we find

$$\boxed{J(x) = \frac{\hbar}{m} \theta'(x) f^2(x) \left(|C_1|^2 - |C_2|^2 \right).}$$

3. The momentum expectation value is

$$\langle \psi | \hat{p} \psi \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \psi' dx.$$

But before we go into calculating this in terms of $f(x)$ and $\theta(x)$, let us recall that we already calculated $\psi^* \psi'$ in terms of the real functions $f(x)$ and $\theta(x)$, and that $\langle p \rangle$ must be *real*! Therefore, the only term that can be in the answer is the imaginary part of $\psi^* \psi'$,

$$\boxed{\langle \psi | \hat{p} \psi \rangle = \hbar \left(|C_1|^2 - |C_2|^2 \right) \int_{-\infty}^{\infty} \theta' f^2 dx},$$

which is just $J(x)$, up to factor of m ,

$$\boxed{\langle \psi | \hat{p} \psi \rangle = m \int_{-\infty}^{\infty} J(x) dx.}$$

It is straightforward that both $J(x)$ and $\langle p \rangle$ vanish when $|C_1| = |C_2|$, i.e. when the wave function $\psi(x)$ is real.

Question 5:

Consider a free particle described by the wave function

$$\Psi(x, t) = \left(A e^{i\frac{p}{\hbar}x} + B e^{-i\frac{p}{\hbar}x} \right) e^{-i\frac{E}{\hbar}t},$$

where $E = p^2/2m$. Find the probability current $J(x, t)$ associated with this wave function. Interpret the different terms and show that if $|A| = |B|$ the probability current vanishes.

Solution:

In order to calculate $J(x, t) = \frac{\hbar}{m} \text{Im} \left[\Psi^* \frac{\partial \Psi}{\partial x} \right]$, we calculate $\Psi^* \frac{\partial \Psi}{\partial x}$,

$$\frac{\partial \Psi}{\partial x} = i\frac{p}{\hbar} \left[A e^{i\frac{p}{\hbar}x} - B e^{-i\frac{p}{\hbar}x} \right] e^{-i\frac{E}{\hbar}t},$$

thus

$$\begin{aligned} \Psi^* \frac{\partial \Psi}{\partial x} &= i\frac{p}{\hbar} \left(A^* e^{-i\frac{p}{\hbar}x} + B^* e^{i\frac{p}{\hbar}x} \right) \left(A e^{i\frac{p}{\hbar}x} - B e^{-i\frac{p}{\hbar}x} \right) \\ &= i\frac{p}{\hbar} \left(|A|^2 + 2i \text{Im} \left[AB^* e^{i\frac{2p}{\hbar}x} \right] - |B|^2 \right). \end{aligned}$$

Therefore

$$J(x, t) = \frac{p}{m} \left(|A|^2 - |B|^2 \right).$$

The interpretation is that the wave function is made out of two states of free a free particle: one propagates to the positive x direction and the other to the negative x direction. Therefore, when both amplitudes are the same we get a cancellation of the probability current.

Question 6:

Consider a particle that is free to move on a ring of circumference L , such that $\Psi(x, t) = \Psi(x + L, t)$.

1. Find the normalized stationary states of the system and explicitly show that they form an orthonormal basis.
2. In the sum $\psi(x) = \sum c_n \psi_n(x)$, what condition must apply on the coefficients c_n so that $\psi(x)$ would be a "legal" quantum state?
3. Show that any linear combination of the stationary states is also a solution of the (time-dependent) Schrödinger equation if $\omega_n = \frac{\hbar k_n^2}{2m}$.
4. Calculate the dispersion relation $\omega_n(k_n)$ and show that $\omega_n = \omega_{-n}$.
5. Given some initial condition $\Psi(x, 0) = f(x)$, where f is some given normalized function, find a way to calculate the coefficients c_n .

Solution:

1. Writing the stationary wave function $\psi(x)$ as a combination of plane waves

$$\psi(x) \propto e^{\pm ikx},$$

we find that

$$\psi(x) = \psi(x + L) \quad \rightarrow \quad e^{ikL} = 1 \quad \rightarrow \quad k \equiv k_n = \frac{2\pi n}{L}.$$

meaning that the stationary states of the system are a discrete set of plane waves

$$\psi_n(x) = N_n e^{ik_n x}, \quad \text{with } k_n = \frac{2\pi n}{L},$$

for which, the normalization factor is

$$\langle \psi_n, \psi_n \rangle = \int_0^L |N_n|^2 dx = N_n L = 1 \quad \text{if} \quad |N_n| = \frac{1}{\sqrt{L}}.$$

2. Considering any linear combination of states,

$$\psi(x) = \sum_n c_n \psi_n(x),$$

the normalization condition states that

$$\langle \psi, \psi \rangle = \sum_{n,m} c_n^* c_m \langle \psi_n, \psi_m \rangle = \sum_{n,m} c_n^* c_m \delta_{nm} = \boxed{\sum_n |c_n|^2 = 1}.$$

3. Since we are dealing with a free particle, the Schrödinger equation reads

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}.$$

Plugging in the time dependent wave function

$$\Psi(x, t) = \sum_n c_n \psi_n(x) e^{-i\omega_n t},$$

we find

$$\hbar\omega\Psi = \frac{\hbar^2 k_n^2}{2m} \Psi \quad \rightarrow \quad \boxed{\omega = \omega_n = \frac{\hbar k_n^2}{2m}}.$$

4. Plugging in the expression for k_n we find

$$\boxed{\omega_n = \frac{2\pi^2 \hbar n^2}{L^2 m} = \omega_{-n}}.$$

5. Considering the initial condition we have

$$f(x) = \sum_m c_m \psi_m(x).$$

Using the orthonormality of ψ_n we can write

$$\boxed{\langle f | \psi_n \rangle = \sum_m c_m \langle \psi_m | \psi_n \rangle = c_n}.$$