Homework 3 - Fourier Transform and Wave Function

Question 1:

Derive the Fourier series for the function $f(x) = x^2$ in the domain $[-\pi, \pi]$. Use this to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution:

Expanding $f(x)$ into Fourier series we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_n c_n e^{inx},$$

where

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 e^{-inx} dx.$$

If $n = 0$ we find

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 dx = \sqrt{2\pi} \frac{\pi^2}{3},$$

if $n \neq 0$ then,

$$\begin{aligned}
c_n &= -\frac{1}{\sqrt{2\pi}} \frac{d^2}{dn^2} \int_{-\pi}^{\pi} e^{-inx} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{d^2}{dn^2} \left[\frac{1}{in} \left(e^{-in\pi} - e^{in\pi} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{dn} \left[\frac{1}{in^2} \left(e^{in\pi} - e^{-in\pi} \right) - \frac{\pi}{n} \left(e^{in\pi} + e^{-in\pi} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{dn} \left[\frac{2}{n^2} \sin(n\pi) - \frac{2\pi}{n} \cos(n\pi) \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\left(\frac{\pi^2}{n} - \frac{2}{n^3} \right) \cancel{\sin(n\pi)} + \frac{2\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n} \right] \\
&= \frac{2\sqrt{2\pi}}{n^2} (-1)^n .
\end{aligned}$$

Therefore, the Fourier series is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \sum_n c_n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2}{n^2} (-1)^n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \left(e^{inx} + e^{-inx} \right),
\end{aligned}$$

or simply

$$\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx)}.$$

Now let us take its value at $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \underbrace{\cos(n\pi)}_{(-1)^n} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \quad \rightarrow \quad \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}.$$

Question 2:

Find the Fourier transform of the Gaussian $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, and show that $\Delta x \Delta k = 1$.

Solution:

Taking the Fourier transform of $g(x)$ yields

$$\begin{aligned}\tilde{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx.\end{aligned}$$

Looking at the power of the exponent we can write

$$-\frac{1}{2\sigma^2}\left(x^2 + 2i\sigma^2 kx\right) = -\frac{1}{2\sigma^2}\left(x^2 + 2i\sigma^2 kx - \sigma^4 k^2\right) - \frac{\sigma^2 k^2}{2},$$

which leads to

$$\tilde{g}(k) = \frac{e^{-\sigma^2 k^2/2}}{2\pi\sigma} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du}_{I},$$

where we defined $u = x + i\sigma^2 k$. we are left with a simple Gaussian integral I which can be solved as follows

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx\right)^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy = \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$$

moving to polar coordinates

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} r dr d\varphi = \pi \int_0^{\infty} e^{-\frac{\xi}{2\sigma^2}} d\xi = 2\pi\sigma^2,$$

where we defined $\xi = r^2$ hence $d\xi = 2rdr$, which gives $I = \sqrt{2\pi\sigma^2}$.
Therefore

$$\boxed{\tilde{g}(k) = \frac{e^{-\sigma^2 k^2/2}}{\sqrt{2\pi}}}.$$

Calculating the square root of the variance $\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ of both results we see that

$$\langle x\rangle = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 0,$$

where the integral vanishes as it is symmetric and the integrand is anti-symmetric.

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

defining $\alpha = \frac{1}{2\sigma^2}$ we can write

$$\begin{aligned}\left\langle x^2 \right\rangle &= -\frac{1}{\sigma\sqrt{2\pi}}\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \\ &= -\frac{1}{\sigma\sqrt{2\pi}}\frac{d}{d\alpha}\sqrt{\pi/\alpha} \\ &= \frac{1}{\sigma 2\sqrt{2}}\left(2\sigma^2\right)^{3/2} \\ &= \sigma^2.\end{aligned}$$

Therefore

$$\Delta x = \sigma.$$

A similar calculation for Δk yields

$$\Delta k = \frac{1}{\sigma},$$

therefore

$$\boxed{\Delta x \Delta k = 1}.$$

Question 3:

Prove the following identity

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] = (ik)^n \mathcal{F}\left[f(x)\right],$$

where $f(x)$ is some analytic function and $\mathcal{F}[]$ denotes a Fourier transform of whatever is in the brackets.

Solution:

Looking at $f(x)$ in terms of the k-space basis,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int \tilde{f}(k) e^{ikx} dk$$

we see that

$$\frac{d^n}{dx^n} f(x) = \frac{1}{\sqrt{2\pi}}\int (ik)^n \tilde{f}(k) e^{ikx} dk.$$

Now, taking the Fourier transform reads

$$\begin{aligned}\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] &= \frac{1}{2\pi}\int\int (ik)^n \tilde{f}(k) e^{ikx} dk e^{-ik'x} dk' \\ &= \int (ik)^n \tilde{f}(k)\, dk \underbrace{\frac{1}{2\pi}\int e^{ix\left(k-k'\right)} dk'}_{\delta(k-k')} \\ &= \left(ik'\right)^n \tilde{f}\left(k'\right),\end{aligned}$$

hence

$$\boxed{\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] = (ik)^n \mathcal{F}\left[f(x)\right]}.$$