Homework 6 - Variables Separation, Infinite Square Well

Question 1:

Following the previous question, consider a particle on a ring that is initially in a state

$$\psi(x) = A\left(\psi_{-2}(x) + \sqrt{2}\psi_{-1}(x) + \psi_0(x) + \sqrt{3}\psi_1(x) + \sqrt{2}\psi_2(x)\right),$$

1. Find $|A|$.
2. Find $\Psi(x,t)$.
3. We define the $\hat{Z}$ operator as

$$\hat{Z}\psi_n(x) = \frac{1+(-1)^n}{2}\psi_n(x).$$

   What is the meaning of this operator? Find the probability for having $\langle Z\rangle = 1$.
4. Find the expectation value for the particle's energy $\left\langle \hat{H}\right\rangle$, show that it is time-independent.
5. Find the probability of measuring $p = 4\pi\hbar/L$.
6. Find the probability of measuring $p = 2\pi\hbar/L$.

   At some time t_0 we measure $\hat{Z}$ of the wave function and get $z = 1$.
7. Write down the wave function $\Psi(x, t > t_0)$. Make sure to normalize it properly.
8. Repeat articles (4) (5) and (6) for the system at times after t_0 .

Solution:

1. Using the normalization condition

$$\langle\psi|\psi\rangle = |A|^2(1+2+1+3+2) = 1 \quad \rightarrow \quad \boxed{|A| = \frac{1}{3}}.$$

2. The time dependent wave function is then

$$\boxed{\Psi(x,t) = \frac{1}{3}\left[\psi_0(x) + \left(\sqrt{2}\psi_{-1}(x) + \sqrt{3}\psi_1(x)\right)e^{-i\frac{E_1}{\hbar}t} + \left(\sqrt{2}\psi_2(x) + \psi_{-2}(x)\right)e^{-i\frac{E_2}{\hbar}t}\right]},$$

   where we defined $E_n = \frac{2\pi^2\hbar^2}{mL^2}n^2$ as the energy of the n'th state.

3. It is easy to see that the eigenvalue of the $\hat{Z}$ operator is 1 if n is even and 0 if it is odd, so we may call it the state "evenness" operator. The probability for measuring $\langle Z\rangle = 1$ is

$$\boxed{\left\langle \Psi|\hat{Z}\Psi\right\rangle = \frac{1}{9}\left(\langle\psi_0|\psi_0\rangle + 2\langle\psi_2|\psi_2\rangle + \langle\psi_{-2}|\psi_{-2}\rangle\right) = \frac{4}{9}}.$$

4. Recalling that the Hamiltonian of a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2},$$

and using the Schrödinger equation

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} \quad \to \hat{H}\Psi = E_n\Psi.$$

Therefore,

$$\boxed{\left\langle \Psi|\hat{H}\Psi\right\rangle = E_n\left\langle \Psi|\Psi\right\rangle = \frac{1}{9}\left(E_0 + 5E_1 + 3E_2\right) = \frac{17}{9}\frac{2\pi^2\hbar^2}{mL^2}}.$$

5. Looking at the eigenvalue of the momentum operator at a certain state

$$\hat{p}\Psi_n = \hbar k_n \Psi_n = \frac{2\pi\hbar n}{L}\Psi_n,$$

we can easily the relation between $\langle p\rangle$ and n, then

$$\boxed{P\left(\langle p\rangle = \frac{4\pi\hbar}{L}\right) = P\left(n=2\right) = \frac{2}{9}}.$$

6. In the same way we have

$$\boxed{P\left(\langle p\rangle = \frac{2\pi\hbar}{L}\right) = P\left(n=1\right) = \frac{3}{9}}.$$

7. Once a measurement was made, the wave function collapses to the state that yields $\langle Z\rangle = 1$, thus

$$\Psi\left(x, t > t_0\right) = N_{t>t_0}\left[\psi_0\left(x\right) + \left(\sqrt{2}\psi_2\left(x\right) + \psi_{-2}\left(x\right)\right)e^{-i\frac{E_2}{\hbar}t}\right],$$

such that

$$\left\langle \Psi\left(x, t>t_0\right)|\Psi\left(x, t>t_0\right)\right\rangle = \left|N_{t>t_0}\right|^2\left(1+3\right) = 1,$$

thus

$$\boxed{\Psi\left(x, t > t_0\right) = \frac{1}{2}\left[\psi_0\left(x\right) + \left(\sqrt{2}\psi_2\left(x\right) + \psi_{-2}\left(x\right)\right)e^{-i\frac{E_2}{\hbar}t}\right]}.$$

8. Repeating the (4-6) we find

$$\boxed{\left\langle \hat{H}\right\rangle_{t>t_0} = \frac{1}{4}\left(E_0 + 3E_2\right) = 3\frac{2\pi^2\hbar^2}{mL^2}},$$

$$\boxed{P\left(\langle p\rangle = \frac{4\pi\hbar}{L}\right) = P\left(n=2\right) = \frac{1}{2}},$$

$$\boxed{P\left(\langle p\rangle = \frac{2\pi\hbar}{L}\right) = P\left(n=1\right) = 0}.$$

Question 2:

Consider a particle of mass m confined in a symmetric one-dimensional potential well of width a

$$V\left(x\right) = \begin{cases} 0, & |x| < \frac{a}{2}, \\ \infty, & \text{else.} \end{cases}$$

1. Find the eigenstates of the Hamiltonian, i.e. stationary states, and the corresponding energies.

2. How does the width of the well a affects the difference between energy levels?

3. Do the eigenstates you found in (1) have a defined parity? How does it depend on n?

4. Find the expectation value for the position of the particle when the system is prepared in a specific eigenstate.

Solution:

1. We have already solved the problem of an infinite square well at $0 < x < a$, and got the stationary states

$$\psi_n(x') = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x'\right), & 0 \le x' \le a, \\ 0 & \text{else,} \end{cases} \qquad \text{with} \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}.$$

Then, all we have to do is to set $x' = x + a/2$, so we get

$$\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right), & -\frac{a}{2} \le x \le \frac{a}{2}, \\ 0 & \text{else,} \end{cases}, \qquad \text{with} \qquad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}}.$$

2. The difference between two sequential energy levels is

$$\boxed{\Delta E = E_{n+1} - E_n = \frac{\hbar^2 \pi^2}{2ma^2}(2n+1) \propto a^{-2}}.$$

3. As we mentioned before, for symmetric potential, even and odd states correspond to odd and even state functions. It can be easily seen in this example, since without the $n\pi/2$ term in the sine, it is an odd function around the origin. Then, by adding a phase of odd multiplication of $\pi/2$ we get a cosine which is even around the origin, whereas even multiplication correspond to phases of multiplication of π which does not change the oddness of the sine function.

$$\begin{aligned} \text{odd } n \quad &\rightarrow \quad \psi(-x) = \psi(x), \\ \text{even } n \quad &\rightarrow \quad \psi(-x) = -\psi(x). \end{aligned}$$

4. If the system was preprepared in a certain state ψ_n then

$$\langle \psi_n | \hat{x} \psi_n \rangle = \int_{-\infty}^{\infty} \psi_n^* x \psi_n dx = \frac{2}{a} \int_{-a/2}^{a/2} x \sin^2\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right) dx,$$

but $\sin^2(x + n\pi/2)$ is even (it is either $\cos^2 x$ or $\sin^2 x$, depends on n being odd or even number), hence the integral vanishes and we get

$$\boxed{\langle \psi_n | \hat{x} \psi_n \rangle = 0},$$

which is what we expect from a symmetric potential around the origin.

Question 3:

Consider an infinite square well of width $2L$, with a particle of mass m moving in it ($|x| < L$). It is given that the particle is in the lowest-energy state at $t < 0$. Assume now that at $t = 0$ the walls of the well move instantaneously such that the well doubles ($|x| < 2L$). This transition of the system does not affect the state of the particle, which remains the same at the moment of the transition.

1. Write down the wave function $\Psi(x, 0)$ (before the transition) and find the corresponding energy of the particle.

2. Write down the wave function of the particle at times $t > 0$. Calculate the probability P_n of finding the particle in an arbitrary eigenstate of the modified system. What is the probability of finding the particle in an odd eigenfunction (with respect to x)?

3. Calculate the expectation value of the energy at any time $t > 0$. You can make use of the series
$$\sum_{\nu=0}^{\infty} \frac{(2\nu+1)^2}{\left[(2\nu+1)^2 - 4\right]^2} = \frac{\pi^2}{16}.$$

4. If we assume that the walls move outwards with a finite speed u, our assumptions should still hold provided that this velocity is much larger than the characteristic velocity of the system at times $t < 0$, i.e. $u \ll v_0$. What is v_0?

Solution:

1. The solution for infinite well of width $2L$ at the lowest energy level ($n = 1$) is
$$\psi(x,0) = \psi_1(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right), \qquad \text{with} \qquad E(t<0) = E_1 = \frac{\hbar^2 \pi^2}{8mL^2}.$$

2. After the transition of the system, the stationary states become
$$\psi_n(x) = \sqrt{\frac{1}{2L}} \sin\left(\frac{n\pi}{4L}x + \frac{n\pi}{2}\right), \qquad \text{with} \qquad E_n = \frac{\hbar^2\pi^2 n^2}{32mL^2} = \frac{1}{4}E(t<0),$$
and the general solution is
$$\Psi(x, t<0) = \sum_n c_n \psi_n e^{-iE_n t/\hbar}.$$
In order to find the c_ns we must use the initial wave function, so that
$$c_n = \int_{-\infty}^{\infty} \psi_1^*(x)\,\psi_n(x)\,dx = \frac{1}{\sqrt{2}L}\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) \sin\left(\frac{n\pi}{4L}x + \frac{n\pi}{2}\right) dx,$$
at this point it is clear that for even ns the integral vanishes (symmetric integral of an odd function), therefore we can already limit ourselves to odd ns ($n \to 2n-1$). Thus
$$\begin{aligned}
c_{2n-1} &= \frac{(-1)^{n+1}}{\sqrt{2}L}\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi}{4L}x\right) dx \\
&= \frac{(-1)^{n+1}}{2\sqrt{2}L}\int_{-L}^{L} \left[\cos\left(\frac{(2n-3)\pi}{4L}x\right) + \cos\left(\frac{(2n+1)\pi}{4L}x\right)\right] dx \\
&= \frac{2(-1)^{n+1}}{\sqrt{2}\pi}\left[\frac{1}{2n-3}\sin\left(\frac{(2n-3)\pi}{4L}x\right) + \frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi}{4L}x\right)\right]\Bigg|_{-L}^{L} \\
&= \frac{2\sqrt{2}(-1)^{n+1}}{\pi}\left[\frac{1}{2n-3}\sin\left(\frac{(2n-3)\pi}{4}\right) + \frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi}{4}\right)\right] \\
&= \frac{2\sqrt{2}(-1)^{n+1}}{\pi}\left[-\frac{1}{2n-3} + \frac{1}{2n+1}\right]\sin\left(\frac{(2n+1)\pi}{4}\right) \\
&= \frac{8\sqrt{2}(-1)^{n}}{\pi(2n-3)(2n+1)}\sin\left(\frac{(2n+1)\pi}{4}\right).
\end{aligned}$$
The probability of finding the particle at some state is
$$\boxed{P_n = |c_n|^2 = \frac{128}{\pi^2 (n-2)^2 (n+2)^2}\sin^2\left(\frac{(n+2)\pi}{4}\right) = \frac{64}{\pi^2 (n^2-4)^2}, \qquad \text{where} \qquad n = 1,3,5,\ldots}$$

3. The expectation value of the energy is
$$\begin{aligned}
\left\langle \hat{H} \right\rangle &= \sum_n |c_n|^2 E_n \\
&= \frac{2\hbar^2}{mL^2}\sum_{\text{odd } n} \frac{n^2}{(n^2-4)^2} \\
&= \frac{2\hbar^2}{mL^2}\sum_{\nu} \frac{(2\nu+1)^2}{\left[(2\nu+1)^2 - 4\right]^2},
\end{aligned}$$

thus

$$\boxed{\left\langle \hat{H} \right\rangle = \frac{\pi^2 \hbar^2}{2m\left(2L\right)^2}}.$$

4. Before the transition, the characteristic velocity of the system was

$$\boxed{v_0 = \sqrt{\frac{2E\left(t<0\right)}{m}} = \frac{\hbar\pi}{2mL}},$$

thus, the walls must move much faster than v_0.

Question 4:

Show that if the function $\Psi\left(x,t\right)$ is a solution of the Schrödinger equation then so is the function $\Psi'\left(x,t\right) = \Psi\left(x,t\right)e^{i\varphi}$ where φ is some arbitrary phase. Do we have to renormalize $\Psi'\left(x,t\right)$ after this transition? How does it affect the probability density $\rho\left(x,t\right)$ and the probability current density $J\left(x,t\right)$?

Solution:

The time-dependent Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi,$$

plugging $\Psi \to \Psi e^{i\varphi}$, changes nothing since

$$i\hbar\frac{\partial}{\partial t}\left(\Psi e^{i\varphi}\right) = e^{i\varphi}i\hbar\frac{\partial}{\partial t}\Psi \quad \text{and} \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\Psi e^{i\varphi}\right) + V\left(\Psi e^{i\varphi}\right) = -e^{i\varphi}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + e^{i\varphi}V\Psi,$$

hence the phase cancels out and the equation remains the same.

Since the norm of the phase $e^{i\varphi}$ is 1, there is no need to renormalize the wave function,

$$\left\langle \Psi | \Psi \right\rangle \quad \to \quad \left\langle \Psi e^{i\varphi} | \Psi e^{i\varphi} \right\rangle = e^{-i\varphi}e^{i\varphi}\left\langle \Psi | \Psi \right\rangle = \left\langle \Psi | \Psi \right\rangle = 1.$$

In the same manner

$$\rho\left(x,t\right) = \left|\Psi\right|^2 \quad \to \quad \left|\Psi e^{i\varphi}\right|^2 = \left|\Psi\right|^2 = \rho,$$

and

$$J\left(x,t\right) = \frac{\hbar}{m}\text{Im}\left[\Psi^*\frac{\partial}{\partial x}\Psi\right] \quad \to \quad \frac{\hbar}{m}\text{Im}\left[\left(\Psi e^{i\varphi}\right)^*\frac{\partial}{\partial x}\left(\Psi e^{i\varphi}\right)\right] = \frac{\hbar}{m}\text{Im}\left[\Psi^*\frac{\partial}{\partial x}\Psi\right] = J\left(x,t\right).$$