Classical Optics

Study of light: Wave / Particle duality of photons

$E = h\nu$ Einstein: Photoelectric Effect $h = 6.63 \times 10^{-34}$ J·s

Wave aspect of light stems from the unification of $\vec{E}$ & $\vec{B}$

Through Maxwell's equations in vacuum:

Gauss's Law:	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$,	$\oiint_A \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_o}\iiint_V \rho\, dV$
Magnetic Flux Law: (i.e. no magnetic monopoles)	$\vec{\nabla}\cdot\vec{B} = 0$,	$\oiint_A \vec{B}\cdot d\vec{S} = 0$
Faraday's Law of Induction:	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$,	$\oint_C \vec{E}\cdot d\vec{l} = -\iint_A \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$
Generalized Circuital (Ampere's) Law:	$\vec{\nabla}\times\vec{B} = \mu_o\vec{j} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$	$\oint_C \vec{B}\cdot d\vec{l} = \mu_o\iint_A \vec{j}\cdot d\vec{S} + \mu_o\varepsilon_o\iint_A \frac{\partial\vec{E}}{\partial t}\cdot d\vec{S}$

In Cartesian coordinates, Maxwell's equations in free space are:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \qquad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Electromagnetic waves

Ohm's law: $\vec{j} = \sigma \vec{E}$

$$\nabla \times (\nabla \times \vec{B}) = \mu\sigma(\nabla \times \vec{E}) + \mu\varepsilon \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla \times (\nabla \times \vec{B}) = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Since

$$\nabla \times (\nabla \times) = \nabla(\nabla \cdot) - \nabla^2$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla \times (\nabla \times \vec{E}) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\nabla \rho$$

For an uncharged medium,

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Equations of telegraphy

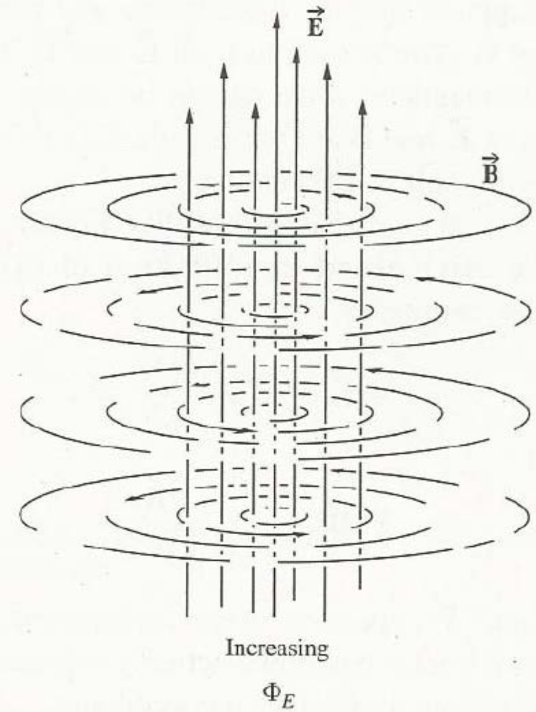


Figure 3.12 A time-varying $\vec{E}$-field. Surrounding each point where Φ_E is changing, the $\vec{B}$-field forms closed loops.

Wave equation $$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$c = v = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$\varepsilon_0\mu_0 = (8.85\times10^{-12}\, s^2C^2 / m^3\cdot Kg)(4\pi\times10^{-7}\, m\cdot Kg / C^2) = 11.12\times10^{-18}\, s^2 / m^2$$

$$c = 3\times10^8\, m / s$$

(as predicted by Maxwell in the year 1861)

Solutions of 1D wave equations

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$\xi = x + vt \qquad \eta = x - vt$$

$$x = \tfrac{1}{2}(\xi + \eta) \qquad t = \tfrac{1}{2v}(\xi - \eta)$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} = \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) = \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)^2\psi = \left(\frac{\partial^2}{\partial\xi^2} + 2\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right)\psi$$

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial t} = v\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\psi$$

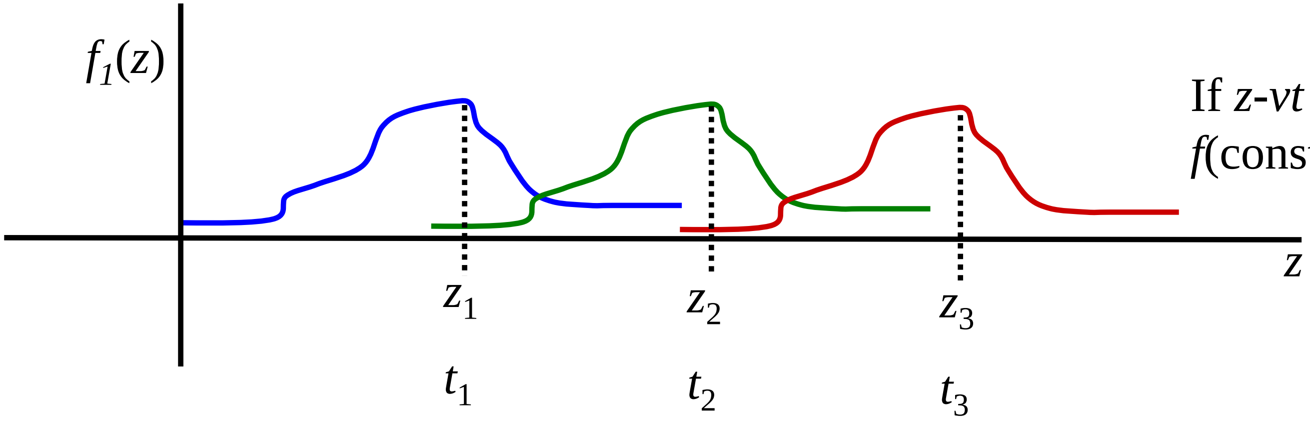
$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right) = \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)^2\psi = v^2\left(\frac{\partial^2}{\partial\xi^2} - 2\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right)\psi$$

Substituting in the wave equation, we obtain $\dfrac{\partial^2\psi}{\partial\xi\,\partial\eta} = 0$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \eta}\right) = 0 \quad \Rightarrow \quad \frac{\partial \psi}{\partial \eta} = F(\eta)$$

$$\psi = \int F(\eta)d\eta + f_2(\xi) = f_1(\eta) + f_2(\xi)$$

$$\psi(x,t) = f_1(x - vt) + f_2(x + vt)$$



If z-vt = const. then
f(const.) = const.

The functions f_1 and f_2 maintain their shape in space and time and move in time in positive and negative directions respectively with velocity v.

In general, the 3D wave equation has the form:

$$\nabla^2\psi(\vec{r},t) = \frac{1}{v^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2}$$

$$\psi(\vec{r},t) = f_1(\vec{r}\cdot\vec{k}/k - vt) + f_2(\vec{r}\cdot\vec{k}/k + vt)$$

One kind of solutions: plane waves.

Spherical coordinates:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial^2 r}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\theta^2}$$

Spherical waves:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

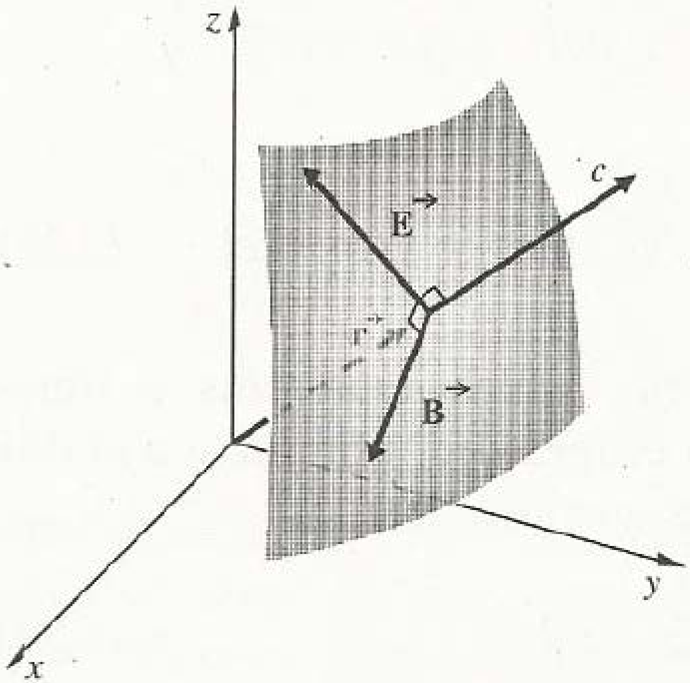


Figure 3.15 Portion of a spherical wavefront far from the source.

We obtain the equation for the 1D wave for $r\psi$

$$\psi(r,t) = \tfrac{1}{r} f_1(r - vt) + \tfrac{1}{r} f_2(r + vt)$$

The second term is usually cancelled from physical reasons.

The envelope of the wave decays like $1/r$ $\rightarrow$ energy is conserved on the envelope $4\pi r^2$.

$$u = u_E + u_B = \varepsilon_0 E^2 + \tfrac{1}{\mu_0} B^2$$

There is an irregularity at r=0. It is resolved if one assumes a source at this point.

Transverse waves

Consider a simple case of a plane wave propagating in vacuum in the positive *x* direction.

$$\vec{E} = \vec{E}(x,t)$$

Since ρ=0,

$$\frac{\partial E_x}{\partial x} = 0$$

E_x that is not dependent on *x* is not a traveling wave advancing in the *x* direction.
Therefore, E_x=0. The *E*-filed is exclusively transverse.
At any moment, we have to specify the direction of *E* – polarization.
Linearly polarized waves:

$$\vec{E} = \hat{j}E_y(x,t)$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

B_x and B_y are constant. The time-dependent **B**-filed can only have a component in the *z* direction.

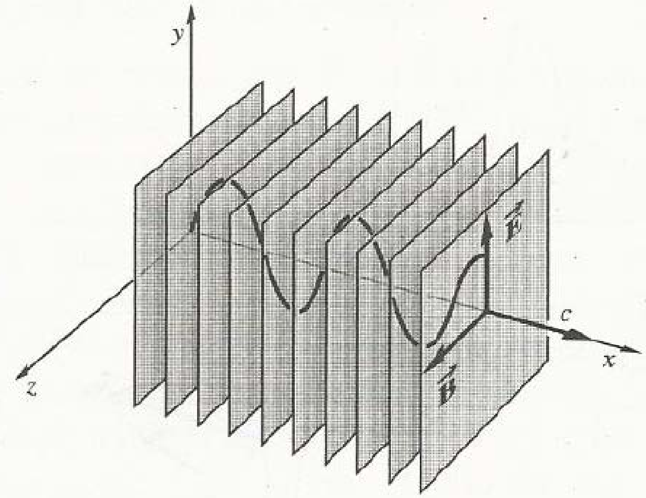


Figure 3.13 The field configuration in a plane harmonic electromagnetic wave.

In real material media the waves are generally not transverse.

Harmonic functions

$$E_y(x,t) = E_{0y}\cos[\omega(t - x/c) + \varepsilon]$$

$$B_z = -\int \frac{\partial E_y}{\partial x}dt = -\frac{\omega}{c}\int E_{0y}\sin[\omega(t - x/c) + \varepsilon]dt = \frac{1}{c}E_{0y}\cos[\omega(t - x/c) + \varepsilon]$$

$$E_y = cB_z$$

$$\psi(\vec{r}) = Ae^{i(\vec{k}\cdot\vec{r}\pm\omega t)} = Ae^{i(2\pi\vec{k}/(k\lambda))\vec{r}\pm\omega t)}$$

E and B are in-phase at all points in space. E and B are mutually perpendicular. Their cross-product, $\vec{E} \times \vec{B}$, points in the perpendicular direction.

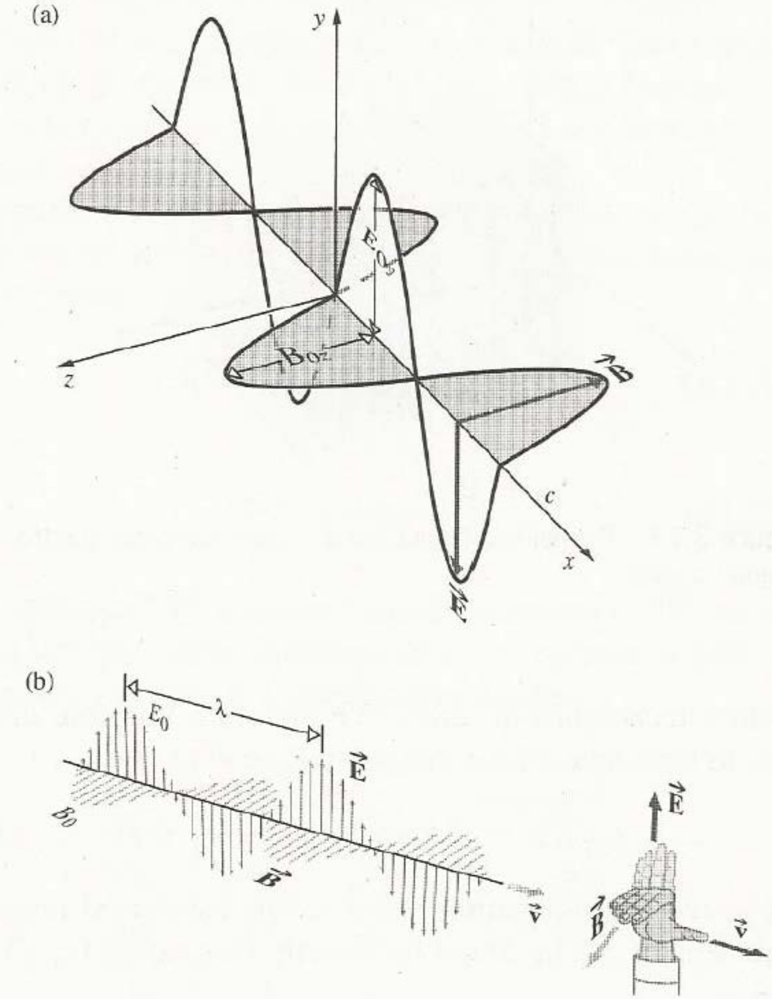


Figure 3.14 (*a*) Orthogonal harmonic $\vec{\mathbf{E}}$- and $\vec{\mathbf{B}}$-fields for a plane polarized wave. (*b*) The wave propagates in the direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$.

Energy and momentum in a propagating wave

The Poynting vector

Energy density of the *E* field: $u_E = \frac{\varepsilon_0}{2} E^2$

Energy density of the *B* field: $u_B = \frac{1}{2\mu_0} B^2$

Therefore, $u_E = u_B$ $\quad u = u_E + u_B = \varepsilon_0 E^2 = \frac{1}{\mu_0} B^2$

Energy flux passing through an area A in time interval Δt: $S = \dfrac{uc\Delta t A}{\Delta t A} = uc$

$$S = \frac{1}{\mu_0} EB$$

Assumption: energy flows in the direction of the propagation of wave (isotropic material).

Definition: $S = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ Poynting vector

$$\vec{S} = c^2 \varepsilon_0 \vec{E} \times \vec{B}$$

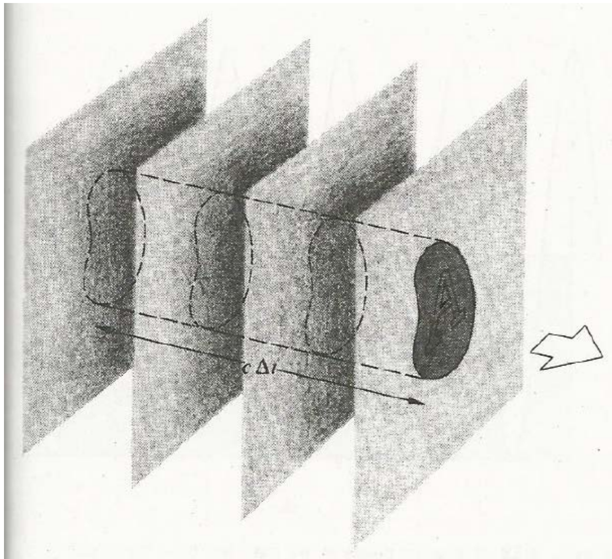


Figure 3.16 The flow of electromagnetic energy.

For harmonic waves:

$$\vec{E} = \vec{E}_0 \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

$$\vec{B} = \vec{B}_0 \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

$$\vec{S} = c^2 \varepsilon \vec{E}_0 \times \vec{B}_0 \cos^2\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

is the instantaneous flow of energy per unit area per unit time

With moving charges:

Development of the idea of E-M wave propagation.

Let's compute $$-\vec{\nabla}\cdot\vec{S} = -\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}/\mu_o\right)$$

Using the differential vector identity:

$$\nabla\cdot(\vec{Q}\times\vec{R}) = \vec{R}\cdot(\vec{\nabla}\times\vec{Q}) - \vec{Q}\cdot(\vec{\nabla}\times\vec{R}) \qquad \vec{Q}=\vec{E},\ \ \vec{R}=\vec{B}$$

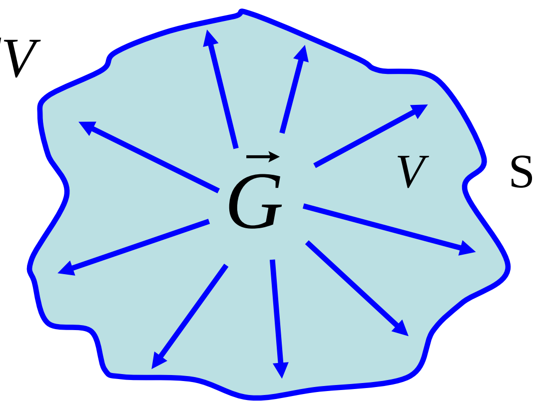
$$-\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}/\mu_o\right) = \tfrac{1}{\mu_o}\left\{\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})\right\} =$$

$$= \tfrac{1}{\mu_o}\left\{\mu_o\vec{j}\cdot\vec{E} + \mu_o\varepsilon_o\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right\}$$

$$= \vec{j}\cdot\vec{E} + \frac{\partial}{\partial t}\left[\frac{\varepsilon_o E^2}{2} + \frac{B^2}{2\mu_o}\right]$$

Take an integral $\int d^3r$ and use the Divergence theorem: $\int_V \vec{\nabla}\cdot\vec{G}\,d^3r = \oint_S \vec{G}\cdot d\vec{s}$

$$\underset{1}{-\oint_S \left(\vec{E}\times\vec{B}/\mu_o\right)\cdot d\vec{s}} = \underset{2}{\int_V \vec{j}\cdot\vec{E}\,dV} + \underset{3}{\frac{\partial}{\partial t}\int_V \left(\underbrace{\frac{\varepsilon_o E^2}{2}}_{u_E} + \underbrace{\frac{B^2}{2\mu_o}}_{u_B}\right)dV}$$



1 Rate at which total energy in *V* increases; Note that $+\oint\left(\vec{E}\times\vec{B}/\mu_o\right)\cdot d\vec{s}$ = rate at which energy flows out of *V* across the boundary S.

2 Rate at which the Kinetic Energy of the particles change.

$$\vec{j} = nq\vec{v} \Rightarrow \int n(\vec{r})\vec{F}(\vec{r})\cdot\vec{v}\,dV$$

Power = Force ×Velocity

3 Rate at which Energy stored in the Fields increase; units in Joules/sec or Watts

Define: $\vec{S} = \vec{E}\times\vec{B}/\mu_o$ Poynting vector; points in the direction in which the fields **E** and **B** transport Energy.

(Units W/m^2)

Complex representation

It is possible to express in 3D using a complex field representation:

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{E}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{E} = \mathrm{Re}\,\vec{\mathcal{E}}$$

$$\vec{\mathcal{B}}(\vec{r},t) = \vec{B}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{B} = \mathrm{Re}\,\vec{\mathcal{B}}$$

With the complex representations, it is possible to derive explicit relations between ***E***, ***B*** and ***k*:**

$$\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \mathrm{Re}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)=0$$

$$\vec{\nabla}\cdot\vec{\mathcal{E}}=i\vec{k}\cdot\vec{E}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow \vec{k}\cdot\vec{E}_o=0 \Rightarrow \vec{k}\perp\vec{E}_o$$

$$also \qquad \vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{k}\cdot\vec{B}_o=0 \Rightarrow \vec{k}\perp\vec{B}_o$$

$$and \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \Rightarrow \mathrm{Re}\left(\vec{\nabla}\times\vec{\mathcal{E}}\right)=\mathrm{Re}\left(-\frac{\partial\vec{\mathcal{B}}}{\partial t}\right)$$

$$\Rightarrow B_o=\frac{1}{\omega}\vec{k}\times\vec{E}_o$$

Let's examine the flow of energy again using the Poynting vector ***S***:

$$\vec{S}=\frac{1}{\mu_o}\vec{E}\times\vec{B} \qquad \vec{E}=E_{yo}\cos(kz-\omega t)\,\hat{j}$$

$$\vec{B}=-E_{yo}\frac{k}{\omega}\cos(kz-\omega t)\hat{i}=\frac{1}{\omega}\vec{k}\times\vec{E}; \quad \vec{B}=\frac{1}{\omega}\vec{k}\times\vec{E}$$

$$\Rightarrow \qquad \vec{S} = -\frac{1}{\mu_o}\cos^2(kz-\omega t)\left(E_{yo}\right)^2\frac{k}{\omega}\hat{j}\times\hat{i}$$

$$= \frac{1}{\mu_o}\cos^2(kz-\omega t)\left(E_{yo}\right)^2\frac{k}{\omega}\hat{k} \qquad c^2 = \frac{1}{\mu_o\varepsilon_o} \qquad \omega = ck$$

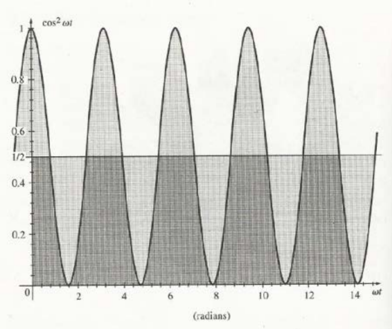
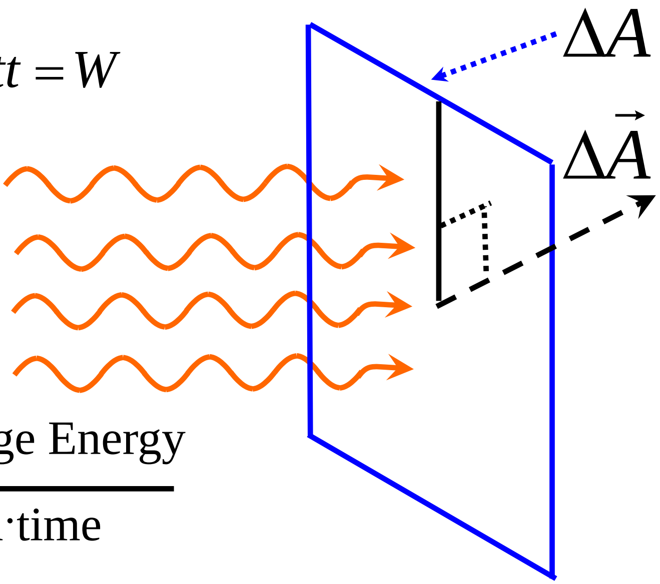


Figure 3.18 Using the peaks above the $\frac{1}{2}$ line to fill the troughs beneath it suggests that the average is $\frac{1}{2}$.

$$because \qquad \left\langle \cos^2(\ldots)\right\rangle_t = \frac{1}{2} \qquad \Rightarrow \left\langle \vec{S}\right\rangle_t = \frac{1}{c}\frac{E_{yo}{}^2}{2\mu_o}\hat{k} = \frac{1}{2}c\varepsilon_o E_{yo}{}^2\hat{k}$$

$$\left\langle \vec{S}\right\rangle \cdot \Delta\vec{A} = \frac{Energy}{Sec.} = \frac{J}{s} = Watt = W$$



Therefore, we can define **irradiance** as

$$I = \left|\left\langle S\right\rangle_t\right| = \frac{c\varepsilon_o}{2}E_o{}^2 = \frac{\text{Average Energy}}{\text{Area·time}}$$

In older texts (and in discussion) the term "**intensity**" is also used.

The energy per unit volume or energy density stored in the fields can be written as before

$$u = u_E + u_B = \frac{\varepsilon_o E^2}{2} + \frac{B^2}{2\mu_o} = \frac{\varepsilon_o E^2}{2} + \frac{E^2}{2\mu_o c^2} = \varepsilon_o E^2 = \frac{1}{\mu_o} B^2$$

Note, again a factor of ½ must be added for the time averages:

$$\langle u \rangle_t = \langle u_E + u_B \rangle_t = \frac{\varepsilon_o E_o{}^2}{2} = \frac{1}{2\mu_o} B_o{}^2$$

Momentum of EM radiation

Thus, similar to the Poynting vector, the E-M momentum *P* per unit volume that exerts a radiation pressure is given by:

$$\vec{P} = \frac{\vec{S}}{c^2}$$

Jackson, 2nd ed., P. 236

$$\vec{P} = \frac{1}{c^2\mu_0}\vec{E}\times\vec{B} = \varepsilon_o\vec{E}\times\vec{B} = \frac{\varepsilon_o}{c}E^2$$

For free space

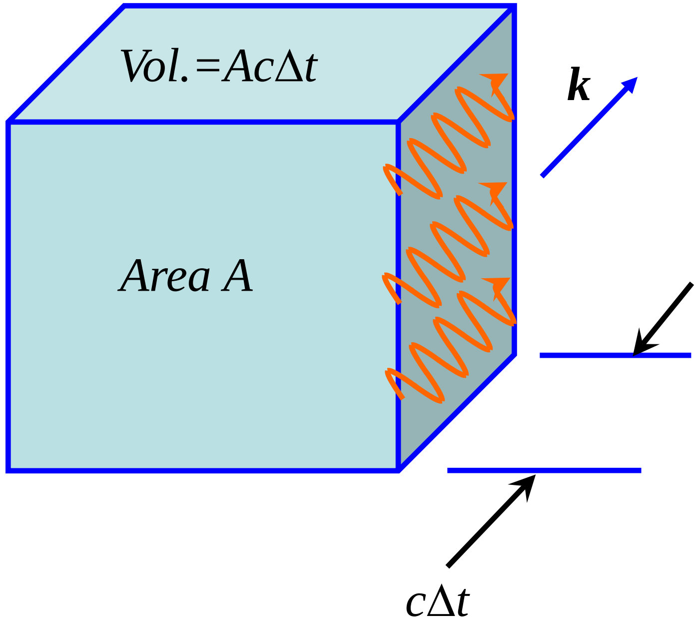
$$\vec{P} = \frac{\varepsilon_o}{c}E_{yo}{}^2\cos^2(kz-\omega t)\hat{k}$$

Harmonic wave

$$\left\langle \vec{P} \right\rangle_t = \frac{1}{2}\frac{\varepsilon_o}{c}E_{yo}{}^2\hat{k}$$

Units: $$\frac{\varepsilon_o E^2}{c} = \frac{J}{m^3}\frac{1}{m/s} = \frac{momentum}{volume}$$

Consider a slab of E-M radiation whose thickness is $c\Delta t$ and cross-sectional area A:



If the light is absorbed by an object, the momentum transfer is given by the impulse force·time:

$$\vec{F} \cdot \Delta t = \Delta \vec{P} = \left| \left\langle \vec{P} \right\rangle_t \right| Ac\Delta t$$

$$\left\langle P_r \right\rangle_t = \frac{F}{A} = \frac{1}{2}\frac{\varepsilon_o}{c} E_{yo}^2 \cdot c = \frac{1}{2}\varepsilon_o E_{yo}^2$$

$$= \left\langle u_E + u_B \right\rangle_t$$

Thus, the energy/vol. contained in the E-M propagation also represents the pressure exerted on an object.

For example, if E_{yo}= 1 V/m, then $\langle P_r \rangle$ = 4.4 x 10^{-12} N/m^2 ≈ 10^{-17} atm.

Note also that $\dfrac{\langle S \rangle}{\langle P_r \rangle} = \dfrac{\frac{1}{c}\frac{E_{yo}^2}{2\mu_o}}{\frac{1}{2}\varepsilon_o E_{yo}^2} = \dfrac{1}{c\mu_o\varepsilon_o} = c$ *or* $I = P_r \cdot c$

For a perfectly reflecting surface, $P_r = \frac{2I}{c}$

The flux density of the electromagnetic energy from the Sun impinging on a surface just outside of the Earth's atmosphare is about 1400 W/m^2. For complete absorption, P_r = 4.7×10^{-6} N/m^2. The Atmospheric pressure is 10^5 N/m^2.
The force on Earth is equivalent to that of 10 tons.