Physics 1C - Tutorial 4

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1 Reminder

1.1 Motion in 2D

Last tutorial, we have seen that if we are dealing with motion in 2D, we can deal with each axis separately since both axes are perpendicular. Therefore, prior definitions hold for each axis. For instance, if we have a position vector:

$$\vec{r} = (r_x\ ,\ r_y) = r_x\hat{x} + r_y\hat{y} \tag{1}$$

Then the velocity can be found by calculating the time derivative of each axis separately:

$$\vec{v} \equiv \dot{\vec{r}} = (\dot{r}_x\ ,\ \dot{r}_y) = (v_x\ ,\ v_y) = v_x\hat{x} + v_y\hat{y} \tag{2}$$

To find the acceleration, all we need to do is calculate the time derivative of $\vec{v}$:

$$\vec{a} \equiv \dot{\vec{v}} = (\dot{v}_x\ ,\ \dot{v}_y) = (a_x\ ,\ a_y) = a_x\hat{x} + a_y\hat{y} \tag{3}$$

2 Question 1

עצם נזרק מעלה בזווית כך שמהירותו בעת הזריקה גדולה פי 5 ממהירותו בשיא הגובה. מצאו את זווית הזריקה מעל האופק.

נתון כי תאוצת הכובד $g = 10\frac{m}{s^2}$, הזניחו חיכוך עם האוויר.

Solution

In this question, we want to find the angle, θ, and we know that the following relation is true:

$$|v_0| \equiv 5|v(t_{h_{max}})| \tag{4}$$

where $t_{h_{max}}$ is the time the object reaches its maximum height.

We can define the initial velocity as $\vec{v}_0 = (v_{0_x}\ ,\ v_{0_y})$. Now, since we know that the object reaches a maximal height, we understand that at $t_{h_{max}}$, its velocity in the vertical axis, $v_y(t)$, must be 0. Therefore:

$$|v_0| = \sqrt{v_{0_x}^2 + v_{0_y}^2} \equiv |5v_x(t_{h_{max}})| \tag{5}$$

In the system, the acceleration acts only along the vertical axis, and so, the velocity in the $\hat{x}$ direction is constant ($v_x(t_{h_{max}}) = v_{0_x}$). Therefore, we can write the previous equation in the following way:

$$\sqrt{v_{0_x}^2 + v_{0_y}^2} = |5v_{0_x}| \quad \rightarrow \quad v_{0_x}^2 + v_{0_y}^2 = 25v_{0_x}^2 \quad \rightarrow \quad v_{0_y}^2 = 24v_{0_x}^2 \quad \rightarrow \quad \frac{v_{0_y}}{v_{0_x}} = \sqrt{24} \tag{6}$$

Now, all we need to do for us to find the throwing angle is use the definition:

$$\tan\theta = \frac{v_{0_y}}{v_{0_x}} = \sqrt{24} \quad \rightarrow \quad \theta = 78.46° \tag{7}$$

3 Question 2

גוף נזרק המהירות של 30 מטר לשניה בזווית של 57 מעלות מעל האופק.
א. היכן ימצא הגוף לאחר 5 שניות.
ב.לאחר כמה זמן יגיע לשיא הגובה ? מהו שיא הגובה.
ג. לאחר כמה זמן יגיע חזרה לגובה ההתחלתי.
ד. מהו טווח הזריקה ?

Solution

1. The first thing we **must** do is define the coordinate system in which we work. As usual, we define $\hat{x}$ to the right and $\hat{y}$ upwards.

To answer this part of the question, we would want to find the object's equations of motion for each of the axes, so we would like to work in cartesian coordinates. We notice that the initial velocity is given in polar coordinates, $\vec{v}_0 = (30\frac{m}{s}\ ,\ 30°)$. Therefore, the first thing we want to do is find the initial velocity in each of the axes:

$$v_{0_x} = 30\cos(57°) = \frac{49}{3}\ \frac{m}{s} \quad , \quad v_{0_y} = 30\sin(57°) = \frac{629}{25}\ \frac{m}{s} \tag{8}$$

Now, once we know the initial velocities and assume that the object's initial position is at the origin, we can write the equations of motion for both axes. In the $\hat{x}$ direction we have motion with constant velocity:

$$x(t) = v_{0_x}t = \frac{49}{3}t \tag{9}$$

and in the $\hat{y}$ direction we have constant acceleration, $g = -10\ \frac{m}{s^2}$, so the equation of motion is:

$$y(t) = v_{0_y}t + \frac{1}{2}gt^2 = \frac{629}{25}t - 5t^2 \tag{10}$$

Now, all that is left to do is to plug in $t = 5s$ and find the location of the object after five seconds:

$$\vec{r}(t) = (x(t)\ ,\ y(t)) \quad \rightarrow \quad \vec{r}(5) = \big(x(5)\ ,\ y(5)\big) = \big(81.65\ ,\ 0.8\big)\ m \tag{11}$$

2. To find when the object reached the maximal height, we need to find the vertical velocity equation $v_y(t)$ and find when it is 0:

$$v_y(t) = \dot{y}(t) = v_{0_y}t + gt \quad \rightarrow \quad v_y(t_{h_{max}}) \equiv 0 = v_{0_y} + gt_{h_{max}} \quad \rightarrow \quad t_{h_{max}} = -\frac{v_{0_y}}{g} = 2.516s \tag{12}$$

Now, to find the maximal height, h_{max}, all that is left to do is plugin $t_{h_{max}}$ into $y(t)$, so we get:

$$h_{max} = y(t_{h_{max}}) = \frac{629}{25}t_{h_{max}} - 5t_{h_{max}}^2 \quad \rightarrow \quad h_{max} = 31.65\ m \tag{13}$$

3. To find when the object has reached its initial height, we want to find the moment that the object reached the ground:

$$y(t_g) \equiv 0 \quad \rightarrow \quad t_g\left(v_{0_y} + \frac{1}{2}gt_g\right) = 0 \quad \rightarrow \quad t_g = 0\ ,\ -\frac{2v_{0_y}}{g} \tag{14}$$

and since we are interested in finding the time it took the object to reach back to the ground, we can eliminate $t_g = 0$, and so we find that the object reached the ground after:

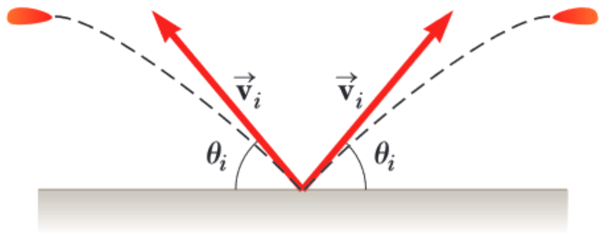
$$t_g = -\frac{2v_{0_y}}{g} = 5.032s \tag{15}$$

4. To find the horizontal distance the object traveled, we need to plug t_g into $x(t)$. And so, we find:

$$x(t_g) = v_{0_x} t_g = \frac{49}{3} \cdot 5.032 \quad \rightarrow \quad x(t_g) = 82.19\ m \tag{16}$$

4 Question 3

שתי טיפות נזרקות בו זמנית ברגע $t = 0$ כמו שנראה באיור. מצאו את המרחק בין הטיפות כפנוקציה של הזמן.
(נלקח משאלות לפתרון עצמי של פרופ' מיכאל גדלין).



Solution

Once again, we define $\hat{y}$ in the upward direction and $\hat{x}$ to the right. Now we would like to find a position vector for each drop.

Drop 1: The initial velocity of the drop is v_i, with angle θ_i. Therefore, the velocity in the x and y axes are $v_x = v_i \cos\theta_i$, $v_y = v_i \sin\theta_i$, respectively. So the position equations are:

$$x_1(t) = v_i \cos\theta_i \cdot t \quad , \quad y_1(t) = v_i \sin\theta_i \cdot t + \frac{1}{2}gt^2 \tag{17}$$

so the position vector of the first drop is:

$$\vec{r}_1(t) = (v_i \cos\theta_i \cdot t\ ,\ v_i \sin\theta_i \cdot t + \frac{1}{2}gt^2) \tag{18}$$

Drop 2: We can follow the same process as for drop 1. However, let's try and approach this differently. We can see that the system is symmetric around the y-axis. This means that the only difference between the position vectors of both drops would be the negative sign of the first component. Therefore:

$$\vec{r}_2(t) = (-v_i \cos\theta_i \cdot t\ ,\ v_i \sin\theta_i \cdot t + \frac{1}{2}gt^2) \tag{19}$$

Now, to find the distance between the drops, all we need to do is find $|\vec{d}(t)| = |\vec{r}_1(t) - \vec{r}_2(t)|$, where $\vec{d}$ is the vector between both drops. Therefore:

$$\vec{d}(t) = \vec{r}_1(t) - \vec{r}_2(t) = (2v_i \cos\theta_i \cdot t\ ,\ 0) \quad \rightarrow \quad |\vec{d}(t)| = 2v_i \cos\theta_i \cdot t \tag{20}$$

Sanity check:
1. The simplest way to check that our answer makes sense is to plugin $\theta_i = \frac{\pi}{2}$ so we find that the distance is always $d = 0$, which makes perfect sense.

2. We can also plugin $\theta_i = 0$ which means that the motion is only on the x-axis, so we get that for a given time, the distance between both drops is $2v_i t$, which intuitively, is precisely what we would think.

5 Question 4

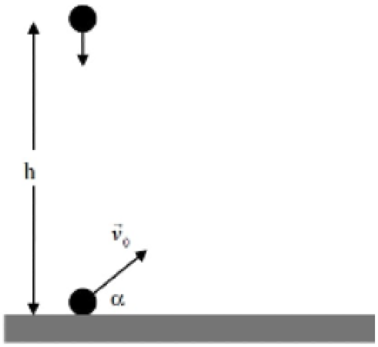
כדור נופל מגובה h נפילה חופשית. באותו הזמן נורה פגז מתותח, הנמצא בדיוק מתחת לכדור, במהירות ובזוית . (בהזנחת גובה התותח)

א) כתוב את משוואות התנועה לכדור ולפגז. כתוב משוואת המסלול עבור פגז.

ב) כתוב את רכיבי המהירות של הפגז ושל הכדור.

ג) מהו הטווח של הפגז?

ד) מהו המרחק המינימלי שיהיה בין הכדור לפגז?



Solution

1. The ball is in free fall, so the equations of motion for the ball are given by:

$$x_{ball}(t) = 0 \quad , \quad y_{ball}(t) = h + \frac{1}{2}gt^2 \tag{21}$$

The cannonball has constant acceleration in the vertical axis and constant velocity in the horizontal axis, so the equations of motion for the cannonball are given by:

$$x_c(t) = v_0 \cos\alpha \cdot t \quad , \quad y_c(t) = v_0 \sin\alpha \cdot t + \frac{1}{2}gt^2 \tag{22}$$

Now, to find the path equation for the cannonball, we basically want to find an equation in which for every given x, we get the height (y) of the cannonball. To do so we can isolate t from $x_c(t)$ and plug it into $y_c(t)$. Doing so we get:

$$t = \frac{x_c}{v_0 \cos\alpha} \quad \rightarrow \quad y_c(x_c) = v_0 \sin\alpha \cdot \frac{x_c}{v_0 \cos\alpha} + \frac{1}{2}g\left(\frac{x_c}{v_0 \cos\alpha}\right)^2 \tag{23}$$

rearranging the equation we get:

$$y_c(x_c) = x_c \cdot \tan\alpha + \frac{g}{2v_0^2 \cos^2\alpha} \cdot x_c^2 \tag{24}$$

2. By calculating the derivatives of Eq.(21) and Eq.(22), we can find the velocities of the ball and cannonball, respectively, for each of the axes. For the ball, we find:

$$v_{ball_x}(t) = \dot{x}_{ball}(t) = 0 \quad , \quad v_{ball_y}(t) = \dot{y}_{ball}(t) = gt \tag{25}$$

For the cannonball, we get:

$$v_{c_x}(t) = \dot{x}_c(t) = v_0 \cos\alpha \quad , \quad v_{c_y}(t) = \dot{y}_c(t) = v_0 \sin\alpha + gt \tag{26}$$

3. We can use the path equation that we found earlier, Eq.(24), and find x_c for which the height of the cannonball is 0. So we can write:

$$y_c(x_c) \equiv 0 \quad \rightarrow \quad x_c\left(\tan\alpha + \frac{g}{2v_0^2 \cos^2\alpha} \cdot x_c\right) = 0 \tag{27}$$

Since we are not interested in $x_c = 0$ solution, we conclude that the total horizontal path that the cannonball passed is:

$$x_{c,total} = -\frac{2v_0^2 \tan(\alpha)\cos^2(\alpha)}{g} \quad \rightarrow \quad x_{c,total} = -\frac{v_0^2}{g} \cdot \sin(2\alpha) \tag{28}$$

Notice, as a quick sanity check, that the sign of the expression is negative, but we remember that also g is negative. Therefore, we see that the vertical path is positive, as one would expect.

4. We know that the vector pointing from the ball to the cannonball, lets say $\vec{d}(t)$, is given by:

$$\vec{d}(t) = \vec{r}_c(t) - \vec{r}_{ball}(t) \quad \rightarrow \quad \vec{d}(t) = (v_0 \cos\alpha \cdot t \ , \ v_0 \sin\alpha \cdot t - h) \tag{29}$$

so we know that at all times the following is true:

$$d^2(t) = |\vec{d}(t)|^2 = (v_0 \cos\alpha \cdot t)^2 + (v_0 \sin\alpha \cdot t - h)^2 = h^2 - 2hv_0 \sin\alpha \cdot t + v_0^2 t^2 \tag{30}$$

Now, to find the minimum distance, we can treat this as a "minimum" problem. Therefore, we want to find the derivative of $d^2(t)$ so we can find the value for which $d^2(t)$ is at minimum:

$$\frac{d}{dt} d^2(t) = \frac{d}{dt}\left(h^2 - 2hv_0 \sin\alpha \cdot t + v_0^2 t^2\right) = 2v_0^2 t - 2hv_0 \sin\alpha \equiv 0 \tag{31}$$

solving for t we find that:

$$t_{min} = \frac{h \sin\alpha}{v_0} \tag{32}$$

Now, we must check if this is a minimum point by calculating the second derivative of $d^2(t)$.

Doing so we find that $d^2(t_{min}) > 0$, indicating that for $t = t_{min}$, $d^2(t)$ is indeed a minimum. Now, all that is left for us to do is to plug t_{min} back into $d^2(t)$ so we find that the minimal distance is:

$$d^2_{min} = d^2(t_{min}) = h^2 cos^2\alpha \quad \rightarrow \quad |d_{min}| = h \cos\alpha \tag{33}$$