Lecture note 4: Fluid Dynamics and Waves

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A. Fluid Dynamics

1. *Viscosity*

Consider a plate floating inside a large tank of liquid (for now, we will assume that the length of the plate L is much smaller than the depth of the pool H, meaning $L \ll H$) as illustrated in Figure 1. Moving the plate at a constant velocity v induces a velocity gradient

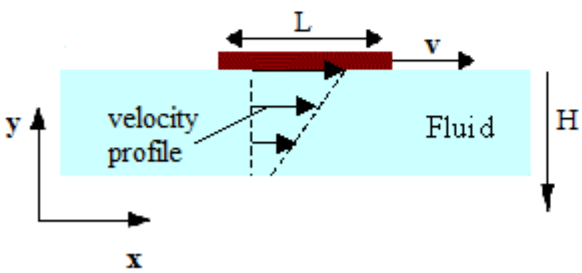

FIG. 1: A plate moving on top of a fluid.

in the liquid below. Each layer of liquid drags the layer below because of viscosity, so there is an downward flux of horizontal momentum. Each layer has an influx of momentum from the above layer and an outwards flux into the layer below. The force per unit area acting on the plate is therefore

$$\frac{F}{S} = \frac{\Delta p}{S\Delta t} = \eta \frac{\partial v_x}{\partial y} , \tag{1}$$

where F is the force applied on the plate for it to move at a constant velocity v, S is its area, $\Delta p/\Delta t$ is the change in momentum over time, v_x is the component of fluid velocity in the direction of the moving plate and η is the dynamic (shear) viscosity. For example, for water the dynamic viscosity is $\eta_{\mathrm{H_2O}} \approx 10^{-3}$ Pa · s.

The energy loss per unit volume is estimated as

$$\dot{\mathcal{E}} = -\frac{Fv}{Sy} = -\eta \left(\frac{\partial v}{\partial y}\right)^2 , \tag{2}$$

which is exact up to a factor of 1/2. Here y denotes the coordinate along which the velocity of the fluid changes, so $\partial v_x/\partial y$ denotes the fluid's velocity gradient.

We can estimate how long it takes for a steady gradient to form. As $\Delta p/\Delta t \sim \eta S v/y$,

$$Ft \simeq mv \quad \longrightarrow \quad \eta \frac{v}{y} St \sim \underbrace{\rho S y}_{m} \underbrace{\frac{v}{2}}_{\text{avg. velocity}}, \tag{3}$$

and therefore $t \sim y^2\rho/2\eta$. This means that after a time t, the gradient reaches down to a depth

$$y \sim \sqrt{2\eta t/\rho} = \sqrt{2\nu t}\,, \tag{4}$$

where $\nu = \eta/\rho$ is the kinematic viscosity.

Notice that for gases, the kinematic viscosity is equal to the diffusion coefficient, $\nu = D = \lambda\bar{v}/3$, where λ is the mean free path and $\bar{v}$ is the thermal velocity. We can estimate ν using $\bar{v} \sim \sqrt{3k_BT/m} \sim 500$ m/s and $\lambda = 1/(\pi d^2 n)$, where d is the molecule diameter and n is the number density of the gas. For air, for example, $\nu_{\text{air}} \sim 1.6 \times 10^{-5}$ m^2/s.

Let us try to estimate the flow velocity of an inclined river. Let us denote by $\alpha \ll 1$ the inclination angle, H the depth of the water, and v_m the river's maximal velocity, as seen in Figure 2.

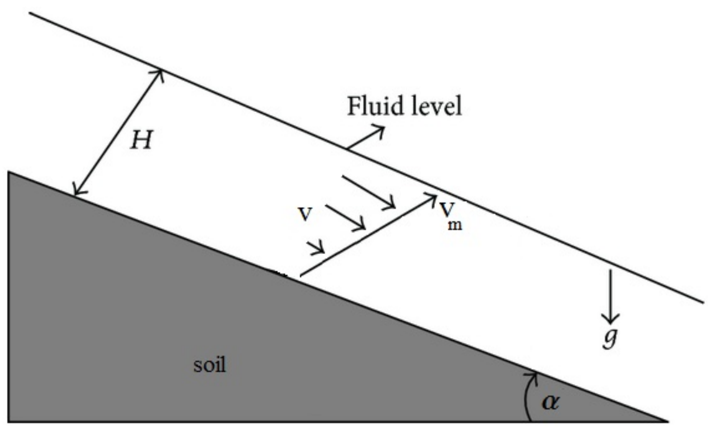

FIG. 2: A cross section of a river

The energy loss is due to viscosity and is balanced by the gravitational energy,

$$\dot{U} = mg\dot{h} = mgv\alpha = mg\frac{v_m}{2}\alpha \stackrel{!}{=} \eta\left(\frac{v_m}{H}\right)^2 SH \sim \nu m\left(\frac{v_m}{H}\right)^2, \tag{5}$$

where S is some unit area of the fluid. This implies that $v \sim v_m/2 \sim gH^2\alpha/4\nu$. Estimating v for a puddle on an inclined road with $H \sim 1$ mm, $\alpha \sim (10\text{ cm})/(10\text{ m}) \sim 10^{-2}$, gives $v \sim 2$ cm/s, which is reasonable. Estimating v for a river with $H \sim 10$ m and $\alpha \sim$

(0.3 m)/(3000 m) $\sim 10^{-4}$ gives $v \sim 2 \times 10^4$ m/s, which is clearly wrong. Can you guess what we did wrong? We will discuss this later on.

2. Drag force

Let us consider a sphere of radius R moving through a fluid with velocity u, e.g. a rain drop falling through air. After a while the a velocity gradient is formed and it is stationary in the sphere frame. Consider spherical shells at a distance r from our sphere. In a steady state, the flux of fluid momentum crossing each shell is constant. This means that $\Delta p/\Delta t = S\eta\nabla v = \text{const.}$, where S is the area of a spherical shell of radius r, and v is the velocity of the fluid. Writing $S \propto r^2$ and treating the flow as spherically symmetric (although it does not really have such symmetry) gives

$$\frac{\partial v}{\partial r} r^2 = \text{const} \quad \longrightarrow \quad v \sim \frac{1}{r}\,. \tag{6}$$

Assuming the sphere with radius R moves with velocity u, we can solve the above equation with the boundary condition of no flow perpendicular to the sphere, and get $v \sim uR/r$. Again, the force per unit area is determined by the velocity gradient, which means:

$$\frac{F_{\text{drag}}}{4\pi R^2} = \eta \frac{u}{R} \quad \longrightarrow \quad F_{\text{drag}} = 4\pi\eta R u\,, \tag{7}$$

this approximation reproduces Stokes' formula, $F_{\text{drag}} = 6\pi\eta R u$.

Lets estimate the velocity of a rain drop using Stokes' formula. There is a force balance between gravity and drag which gives

$$mg = F_{\text{drag}} \quad \longrightarrow \quad u \sim \frac{mg}{6\pi\eta R}\,. \tag{8}$$

Estimating the diameter of a drop to be $d = 3$ mm we get $v \sim 200$ m/s, which is, again, clearly wrong. The following section will give an explanation about the origin of our error.

3. Skin-layer

Looking back at our first problem of a plate in a pool we notice, as shown in Eq. (4), that the velocity profile should reach the maximum depth of $y_{\text{max}} = \sqrt{2\nu t_{\text{max}}} \sim \sqrt{2\nu L/u}$, where L is the plates length. This length scale is much smaller than the pool's depth, $L \ll H$.

Hence, the characteristic length is not the pool depth but rather the maximum depth of the profile, or the 'skin-layer'.

In the case of a spherically symmetric drop with diameter d, the same argument indicates a skin-layer of roughly $\delta \sim \sqrt{2\nu d/u}$. In the case where $\delta \gg d$, there is no skin-layer effect and the information about the moving sphere spreads to infinity. In the case where $\delta \ll d$, the velocity gradient is not u/R but rather u/δ.

Let us calculate the ratio δ/d:

$$\frac{\delta}{d} = \sqrt{2\nu\frac{1}{ud}} = \sqrt{\frac{2}{\mathrm{Re}}} \sim \mathrm{Re}^{-1/2}\,, \tag{9}$$

where Re is Reynolds number. This is useful since we can decide whether skin-layer effects are important or not merely by estimating Re.

In the case of a rain drop, $\mathrm{Re} \sim 200 \times 3 \times 10^{-3}/1.6 \times 10^{-4} \sim 2 \times 10^4 \gg 1$, which means that in order to get a correct free fall velocity we must consider the skin-layer. Correcting our calculations by taking the gradient to be u/δ, we get

$$F_{\mathrm{drag}} = 4\pi R^2 \eta \frac{u}{\delta} \simeq \pi d\eta u\sqrt{\frac{ud}{\nu}}\,. \tag{10}$$

By comparing gravity to drag as above, it follows that the terminal velocity of the drop is $u \sim 15$ m/s, which is much more reasonable than our earlier estimate.

Another way of viewing drag is by inspecting the force a body must exert on the surrounding fluid particles it encounters on its way. The momentum that a spherically shaped droplet transfers to the environment is

$$\Delta p = \underbrace{Su\Delta t}_{\text{Volume covered in time } \Delta t}\rho u\,, \tag{11}$$

so the force is

$$F = \frac{\Delta p}{\Delta t} = S\rho u^2 \propto u^2\,, \tag{12}$$

proportional to u^2. However, this is not true in ideal fluids, for which the same momentum the sphere transfers forward, it gains from the back. We can understand this by approximating the motion of a sphere as it being composed of a series of spheres, each inflating as the previous sphere deflates slightly behind it. The momentum flux is conserved over the area of a sphere, so $v(r) \propto 1/r^2$, like the electric field of a point charge. These velocity fields are analogous to an electric dipole, which is symmetric along its axis. `[There are quite a few spelling errors. Please run a spell checker.]`

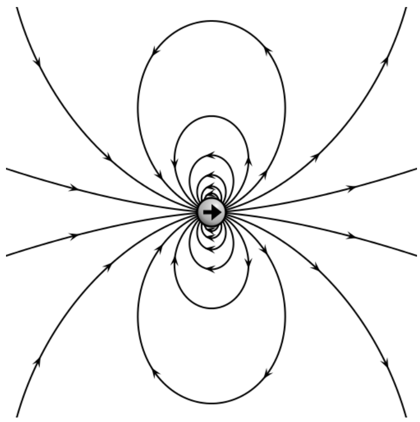

FIG. 3: The velocity field generated by a moving sphere

In this case, the net momentum is constant: the momentum given to the fluid is transferred to the back of the sphere, hence $\vec{F} = 0$. However, experimental results show that for large Reynolds numbers, the force actually is proportional to u^2. We'll try to understand this in the following section.

4. Navier-Stokes Equation

The equation we'll provide (without proof) is actually Newton's second law for an element of fluid which moves with velocity $\vec{v}$,

$$\rho \frac{d\vec{v}}{dt} = -\nabla P + \eta \nabla^2 \vec{v} + \rho \vec{g} \tag{13}$$

Some intuition: if there is no pressure gradient, there would be no momentum change, so we have the first derivative of P. The viscosity term involves a second derivative of $\vec{v}$ because the fluid element experiences two forces (each of them proportional to the first derivative of velocity) in opposite directions - one from each side of the fluid element. The last term gives the gravitational acceleration, $d\vec{v}/dt = \vec{g}$.

Eventually we would like to obtain an expression for $\vec{v}(\vec{r}, t)$. The velocity of the moving element of fluid changes both with time and with position,

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial \vec{r}} d\vec{r} \tag{14}$$

and

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \frac{\partial \vec{v}}{\partial \vec{r}}, \tag{15}$$

so using

$$\vec{v}\frac{\partial \vec{v}}{\partial \vec{r}} = v_i \frac{\partial v_j}{\partial x_i} = (\vec{v}\cdot\nabla)\vec{v}\,, \tag{16}$$

we conclude that

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho} + \nu\nabla^2\vec{v} + \vec{g}\,. \tag{17}$$

This equation has numerous consequences. For example, we can deduce that winds must exist. Indeed, consider a planar atmosphere, with gradients only in the z direction. If we assume that $\vec{v} = 0$ (mechanical equilibrium), we obtain hydrostatic equation $\nabla P = \rho\vec{g}$, where for an ideal gas $P = RT\rho/M$. Here R is the ideal gas constant, T is the temperature, ρ is the density and M is the fluid's molar mass. From the hydrostatic equation it follows that $P = P(z)$ and $\rho = \rho(z)$, so $T = T(z)$. This means that every point with the same altitude in the atmosphere has the same temperature, which we know isn't true. This means that the atmosphere must not be static, i.e. winds must exist.

The Navier-Stokes equation is non-linear. In such equations, the solutions are usually unstable. For a small change in the initial conditions, the solutions will diverge from each other by the Lyapunov exponent, $e^{\lambda t}$, where λ^{-1} is the characteristic time in the problem. For winds on earth, the characteristic time will be

$$T \sim \frac{R_\oplus}{v_{\rm wind}} \sim \frac{10^7 \text{ m}}{10 \text{ m/s}} \approx 10 \text{ days}\,. \tag{18}$$

This time is the upper limit for a weather forecast - For larger times, small perturbations make it impossible to solve the equations.

5. *The Reynolds Number*

We can now define the Reynolds number formally using the Navier-Stokes equation, as the ratio between the nonlinear inertial term and the viscous term,

$$Re \equiv \frac{(\vec{v}\cdot\nabla)\vec{v}}{\nu\nabla^2\vec{v}} \sim \frac{\rho v v/L}{\eta v/L^2} = \frac{\rho v L}{\eta} = \frac{vL}{\eta}\,, \tag{19}$$

where L is some characteristic length scale, v is some characteristic velocity, and $\rho\,,\nu$ and η are the fluid density, dynamic viscosity and kinematic viscosity, respectively.

For $Re \gg 1$, we can neglect the viscosity and treat the problem as flow in an ideal fluid. In such a fluid, unlike our previous description, the flow is not forward-backward symmetric, and the sphere doesn't acquire momentum from its back. Instead, turbulence forms behind it, and the momentum which is transferred from the forward direction is directed to the turbulent region.

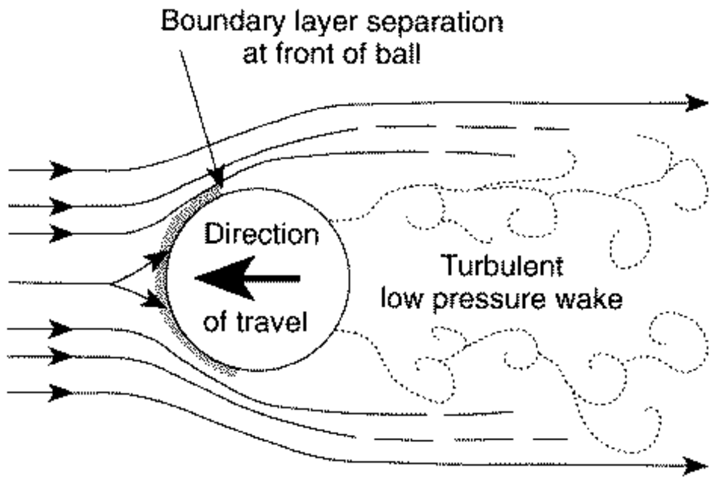

FIG. 4: Illustration of a flow past a sphere [Ref]

Using the same considerations as above, we conclude that the drag force in the $Re \gg 1$ regime is $F_D = \rho S u^2$, so the terminal velocity of a droplet is $u \sim 5$ m/s, which is approximately the correct result.

Airplanes use the same principle to fly - turbulence forms behind the lower edge of the wing, and momentum is transferred to the air molecules in this regime. As a result, the plane acquires momentum in the opposite direction - up into the air.

Let's calculate the ratio between the drag force we've found and the Stokes force,

$$\frac{F_D}{F_S} = \frac{\rho S u^2}{6\pi\eta R u} = \frac{Ru}{6\nu} \sim Re\,. \tag{20}$$

We may define an effective kinematic viscosity, $\nu^* = Ru/6$, such that for $\nu = \nu^*$ these forces are equal, $F_D = F_S$.

We note that in gases, the kinematic viscosity, $\nu = \lambda\bar{v}/3$, is the same as the effective kinematic viscosity ν^* up to a numerical factor. The characteristic length and velocity scales are the mean free path and the thermal velocity.

Now we return to the flowing river problem. We use the same expression for velocity as before, but replace ν by the effective kinematic viscosity,

$$v = \frac{\alpha g H^2}{4\nu^*}\,,\ \nu^* = \frac{1}{6}Hv \quad \longrightarrow \quad v = \sqrt{\alpha g H} = \sqrt{10^{-4}\cdot 10\cdot 10} = 0.1 \text{ m/s}\,, \tag{21}$$

which now is reasonable.

B. Waves and the Fourier Transform

1. Fourier Transform

The Fourier transform is useful in problems which deal with small deviations from equilibrium in a uniform medium. Harmonic functions describe well the linear behavior in such problems, and are much more convenient to work with.

Consider a uniform piece of solid. A force is applied at a point $\vec{r'}$. What is the displacement at $\vec{r}$? Close to equilibrium, we expect a linear behavior which can be described by a matrix $A(\vec{r}-\vec{r'})$, so the expression for the displacement is

$$\vec{u}(\vec{r}) = \int A(\vec{r}-\vec{r'})\vec{F}(\vec{r'})d\vec{r'}\,. \tag{22}$$

In Fourier space, the convolution between the two functions is just a multiplication of the transforms,

$$\vec{u}(\vec{k}) = A(\vec{k})\vec{F}(\vec{k})\,. \tag{23}$$

The energy of the system near equilibrium can be written as

$$U = \int G(\vec{r}-\vec{r'})h(\vec{r})h(\vec{r'})d\vec{r}d\vec{r'}\,. \tag{24}$$

In Fourier space this takes the form

$$U = \sum_{\vec{k}} G(\vec{k}) \left| h(\vec{k}) \right|^2 , \tag{25}$$

where $G(\vec{k})$ can be expanded to even powers (since the medium is isotropic),

$$G(\vec{k}) = \underbrace{G(\vec{k}=0)}_{\text{describes pendulum}} + \underbrace{C_2 k^2}_{\text{acoustic waves}} + \underbrace{C_4 k^4}_{\text{transverse waves in an elastic rod}} + \ldots \tag{26}$$

2. Acoustic Wave Expansion

An acoustic wave is a perturbation in pressure, which implies that along the wave there is a change of pressure and density. Therefore the group velocity c must depend on $\partial P/\partial \rho$, which has units

$$\left[\frac{\partial P}{\partial \rho}\right] = \frac{N \cdot m^3}{m^2 \cdot Kg} = \frac{m^2}{s^2}\,, \tag{27}$$

so by dimensional analysis we get

$$c \sim \sqrt{\frac{\partial P}{\partial \rho}}\,. \tag{28}$$

We recall that Young's modulus E has units of pressure, so in solids

$$c \sim \sqrt{\frac{E}{\rho}}\,. \tag{29}$$

If we go back to the problem of a stone hitting a wall, we now see that we could have estimated the duration of a collision as the time takes an acoustic wave to expand in the stone, $\Delta t \sim l/c$.

In ideal gases $P = RT\rho/M$, where R and M are the gas constant and molar mass, so we might think that

$$c = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{RT}{M}}\,. \tag{30}$$

However, this speed is lower than the one we get in experiments. The reason is that the acoustic expansion is not isothermal, but rather adiabatic. We should use the isentropic process equation $P/\rho^\gamma = \text{const}$, which will provide the constant that gives the correct result $c = (\Gamma P/\rho)^{1/2}$. Here $\gamma = C_P/C_V$ is the adiabatic index (the heat capacity ratio), P is the pressure, and ρ is the density.

3. Gravitational Capillary Waves

We want to calculate the dispersion relation for waves on the surface of deep water. We'll assume a harmonic standing wave (the dispersion relation is similar to a traveling one).

The gravitational energy of an element of water is

$$U_g = \frac{1}{2}\underbrace{L_x L_y h\rho}_{M} gh = \frac{1}{2}L_x L_y \rho g h^2\,. \tag{31}$$

In addition there is an energy from the surface tension. The ratio between the perturbed and unperturbed surface areas is, as illustrated in Fig. 6,

$$\frac{\Delta S}{S_0} \simeq \frac{\sqrt{\left(\frac{\lambda}{4}\right)^2 + h^2} - \frac{\lambda}{4}}{\lambda/4} \underset{h \ll \lambda}{\approx} \frac{8h^2}{\lambda^2}\,. \tag{32}$$

The addition to the surface tension energy is

$$U_\gamma = \gamma L_x L_y \frac{8h^2}{\lambda^2}\,, \tag{33}$$

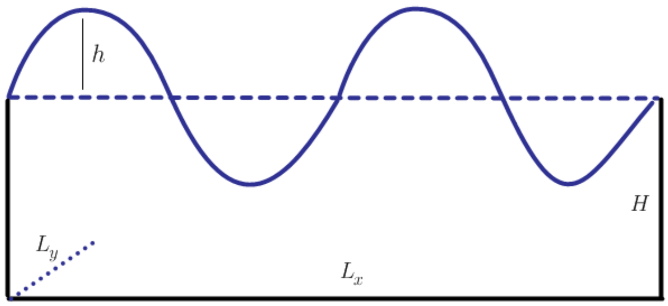

FIG. 5: Waves in a pool of water

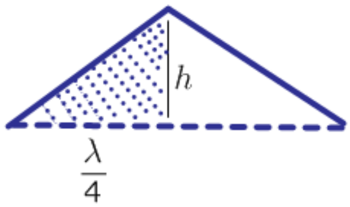

FIG. 6: An approx. of half of a wavelength. The solid and dashed lines are the surface of the water, and the unperturbed surface, respectively. The difference between them is ΔS.

where γ is the surface tension.

In order to calculate the kinetic energy, we claim that the characteristic velocity is $\dot{h}$, and the depth relevant for the motion is in the order of magnitude of λ, rather than h, which is too small to confine all the motion within it.

$$U_k = \frac{1}{2}\underbrace{\lambda L_x L_y \rho}_{M}\dot{h}^2\,. \tag{34}$$

The total energy is

$$E = \frac{1}{2}\underbrace{\lambda L_x L_y \rho}_{M}\dot{h}^2 + \frac{1}{2}\underbrace{\left(\frac{1}{2}L_x L_y \rho g + \gamma L_x L_y \frac{16}{\lambda^2}\right)}_{K}h^2\,. \tag{35}$$

Replacing $k = 2\pi/\lambda$ and using the exact numerical factors we get

$$\omega^2 = \frac{K}{M} = gk + \frac{\gamma}{\rho}k^3 = gk\left(1 + l_c^2 k^2\right)\,, \tag{36}$$

where

$$l_c^2 = \frac{\gamma}{\rho g} \tag{37}$$

is the capillary length. For water $\gamma = 70 \times 10^{-3}$ N/m and $l_c = 2.5$ mm.

The phase velocity can also be calculated as

$$v_\varphi^2 = \frac{\omega^2}{k^2} = g l_c \left(\frac{1}{k l_c} + k l_c \right) . \tag{38}$$

This function has a minimum at

$$v_{\varphi,min} = \sqrt{2 g l_c} \, . \tag{39}$$

For water we find that $v_{\varphi,min} \approx 20$ cm/s. This means that winds or other perturbations slower than ~ 20 cm/s will not excite such waves, see Figure 7.

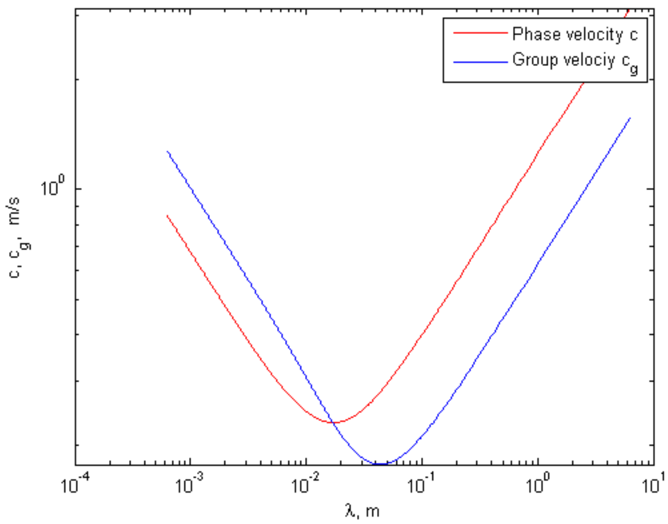

FIG. 7: Group and phase velocity of waves in water vs. wavelength. From the web [Ref].

The dispersion relation tells us the regime where capillary effects are important and where they aren't. For $k \ll l_c^{-1}$ we have $\omega^2 \sim gk$, meaning that $u \sim \sqrt{\lambda}$. For $k \gg l_c^{-1}$ we have $\omega \sim k^{3/2}$ meaning that $u \sim \lambda^{-1/2}$. Deep and shallow gravitational waves are discussed in the next lecture.

B. Energy Losses in Gravitational Waves

In this section we are going to discuss the energy losses in waves on shallow and deep water reservoir. In addition, at the last part of this section, we wish to explain why pouring oil into raging sea water makes the water layer calm. This is what the sailors used to do in a bay for entering into the land peacefully.

1. deep water reservoir

Deep Water Reservoir implies that the wavelength of the wave is much less than the depth of the reservoir, i.e., $\lambda << H$. We wish here to find the decay rate or the rate of loss of Energy, denoted by $\Gamma(k) = \frac{\dot{E}}{E}$.

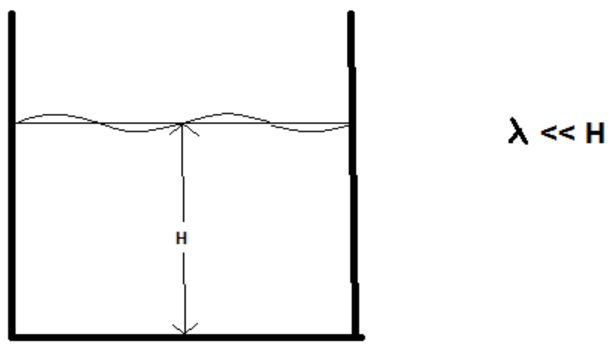

FIG. 3: The deep water reservoir

For gravitational capillary waves, the Potential and Kinetic Energies we had previously derived. The Potential Energy

$$U_p = \frac{1}{2} L_x L_y \rho g h^2$$

and the latter one

$$E_k = \frac{1}{2} L_x L_y \rho \lambda \dot{h}^2.$$

Energy Conservation principle requires that

$$E_P + E_K = C.$$

So, the time derivative will be zero

$$\dot{E}_P + \dot{E}_K = 0$$

which will lead to

$$\ddot{h} + \frac{4gh}{\lambda} = 0.$$

So

$$\omega^2 = \frac{4g}{\lambda} \sim gk$$

or

$$\omega = \sqrt{gk}.$$

For a deep water reservoir, the energy losses due to viscosity and velocity is (as we already saw)

$$\dot{E} = \eta \left(\nabla v\right)^2 L_x L_y \lambda$$

and $\nabla v = \frac{\dot{h}}{\lambda}$, therefore

$$\dot{E} = \eta \left(\frac{\dot{h}}{\lambda}\right)^2 L_x L_y \lambda.$$

The rate of energy loss is

$$\Gamma = \frac{\dot{E}}{E} = \frac{\dot{E}}{2E_k} \approx \frac{\eta}{\rho} k^2 = \nu k^2.$$

where ν is the kinematic viscosity.

2. Shallow water reservoir

Shallow water reservoir means that the depth of the reservoir must be much less than the wavelength of wave, i.e., $H \ll \lambda$.

The potential energy here is same as that for the deep water reservoir

$$E_P = \frac{1}{2} L_x L_y \rho g h^2$$

But the kinetic energy is different here because the typical velocity changes. In the shallow water the particles do not have the depth to plunge into because they would encounter the bottom, the velocity at the bottom is much faster than the plunging velocity and is equal to $\frac{\dot{h}\lambda}{H}$. The depth of plunging is now H, and so the kinetic energy is

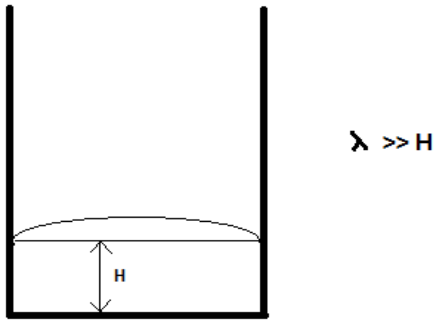

FIG. 4: The shallow water reservoir

$$E_k = \frac{1}{2} L_x L_y \rho H v^2 = \frac{1}{2} L_x L_y \rho H \left(\frac{\dot{h}\lambda}{H}\right)^2 = \frac{1}{2H} L_x L_y \rho \dot{h}^2 \lambda^2.$$

Still Energy Conservation principle should hold here i.e ,

$$E_P + E_K = C.$$

So , the time derivative will be zero , or :

$$\dot{E}_g + \dot{E}_K = 0$$

which will yield

$$\ddot{h} + \frac{4gH}{\lambda^2} h = 0$$

and therefore

$$\omega^2 = \frac{4gH}{\lambda^2} \sim gHk^2$$

or

$$\omega = k\sqrt{gH}$$

$\omega \approx \sqrt{gH}k$ implies there is no dispersion (stationary picture).

The velocity gradient $\frac{\dot{h}}{H}$ is influenced by the fact that the bottom layer is stationary. The time it takes the gradient to stabilize and for the flow to understand that the bottom is not moving is $\frac{H^2}{\nu}$ and it must be larger than the time period of the wave itself .

$$\frac{H^2}{\nu} \gg T \approx \frac{1}{\omega}$$

where T is the wave period time.

So the only layers that are influenced by the gradient are those with

$$\frac{\delta^2}{\nu} \approx \frac{1}{\omega}$$

or

$$\delta \approx \sqrt{\frac{\nu}{\omega}}.$$

The energy loss is

$$\dot{E} = \eta \left(\frac{v}{\delta}\right)^2 L_x L_y \delta \propto \frac{1}{\delta}$$

impling that if the layer δ is small, then the energy loss is great.

Therefore, the rate of energy loss will be

$$\Gamma(k) = \frac{\dot{E}}{E} = \frac{\dot{E}}{2E_k} = \frac{\eta}{\rho H \delta} = \frac{\nu}{\delta H} = \frac{\sqrt{\nu\omega}}{H}.$$

From this we can answer our question as we wished in the beginning of this subsection .

The oil is a hydrophobic substance and therefore the oil particles do not want to detach from the other oil particles . This causes the oil to always float on the upper surface leading to the velocity of the water at the upper surface and at the bottom to equalize (almost zero at the upper surface), producing a the layer of δ . Now, since $\delta \to 0$, the energy loss being inversely proportional must be great, resulting the calmness of the water level.

C. Quantum Mechanics

In this section we wish to derive some of our known quantum relations in the spirit of the Back of Envelope Physics .

1. *Deriving the Schrödinger equation*

The wave equation is

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

if we put $\psi \sim e^{i(\omega t - kx)}$ we would get the dispersion relation

$$\omega = ck$$

meaning that the particles behave as photons. The energy of a photon is $E = \hbar\omega$, the momentum $p = \hbar k$ and the particles energy, for relativistic particles, is $E^2 = m_0^2c^4 + p^2c^2$.

If we put the photon's energy and momentum into the relativistic energy momentum and we would replace, for convenience, $m_0^2c^4 \to \hbar^2\omega_0^2$ we would get the dispersion relation

$$\omega^2 = \omega_0^2 + c^2k^2$$

this resembles a system of coupled harmonic oscillators which are also coupled to pendulums, with ω_0 as the pendulums frequency. This dispersion relation can be achieved from the differential equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\omega_0^2}{c^2}\psi = 0\ .$$

This equation is called the Klein-Gordon equation.

If we insert $\psi(x,t) = \Psi(x,t)\,e^{i\omega_0 t}$ into the Klein-Gordon equation and take the non-relativistic limit ($c \to \infty$) we would get the *Schrödinger* equation

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{2im}{\hbar}\frac{\partial\Psi}{\partial t} = 0\ .$$

2. *Quantum Harmonic Oscillator*

We all remember from the quantum 1 lessons how difficult it is to get the energy levels of quantum harmonic oscillator particles. In order to solve this problem on the back of an envelope, we can look at its potential energy and see that it is quadratic (parabola in 1D), If we approximate this as almost a box then we can solve this problem as if it was a particle in a box, a problem we can solve easily.

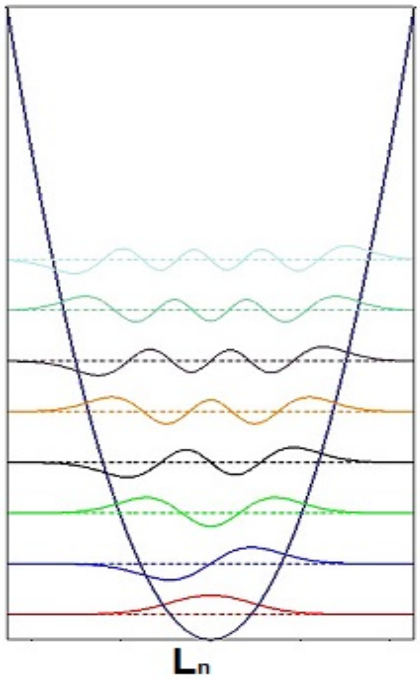

FIG. 5: The Harmonic Oscillator approximated with a Potential Well.

For a particle in a 1D box we know that for a box of size L the wavelength is

$$\lambda_n = \frac{2L}{n}$$

then

$$p_n = \hbar k_n = \frac{2\pi\hbar}{\lambda_n} = \frac{\hbar\pi n}{L}$$

and the energy levels are

$$E_n = \frac{p_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

But now for the harmonic oscillator we can say that the size of box is also n depended $L = L_n$

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL_n^2}.$$

To find what L_n is, we use the fact that the averaged potential energy is equal to the averaged kinetic energy. Again, we can think of it as a particle connected by 2 identical springs at the middle of the box. In this situation, as we know that the "deviation from equilibrium" is $\frac{1}{2}L_n$ and each spring constant is half the spring constant if there was only one spring in the system.

Putting these two situation in the form of equation:

$$\frac{p^2}{2m} = \frac{1}{2}\frac{1}{2}m\omega^2\left(\frac{L_n}{2}\right)^2$$

or

$$\frac{\left(\frac{\hbar\pi n}{L_n}\right)^2}{2m} = \frac{1}{2}\frac{1}{2}m\omega^2\left(\frac{L_n}{2}\right)^2$$

therefore

$$L_n^2 = \frac{\sqrt{8}\pi\hbar n}{m\omega}$$

so that

$$E_n = U_p + E_k = \frac{\left(\frac{\hbar\pi n}{L_n}\right)^2}{2m} + \frac{1}{2}\frac{1}{2}m\omega^2\left(\frac{L_n}{2}\right)^2 = \frac{\pi}{\sqrt{8}}\hbar\omega n \approx \hbar\omega n.$$

If we remember that the exact solution is

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

So for $n \gg 1$ this leads to the same Energy eigenvalues as the original.