LECTURE 1: GUESSTIMATES AND FERMI QUESTIONS

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A. Introduction

1. Goals and Motivation

Our main goal is to acquire a broad and useful picture of physics, and to develop the ability to estimate roughly (up to an order of magnitude) and quickly the answers to various physical questions with a minimum amount of formalism - estimates which can be done in one's head or on the back of an envelope.

The main reasons for developing rough estimates are:

- The underlying physical principles are more transparent and easier to communicate
- Most of our ideas (usually $\gtrsim 90\%$) don't work, it's crucial to dismiss them quickly
- To follow a good idea, important vs. negligible processes must be quickly distinguished
- Important verification of lengthy calculations
- It's fun!

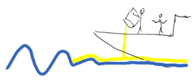
2. Sample Problems

Arbitrary sample of questions we will tackle:

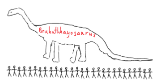
1. Who jumps higher - a kangaroo or a flea?

2. What is the maximal wind speed?

3. Why does pouring oil stem ocean waves?

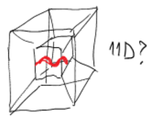


4. What is the upper limit on an animal's size?



5. What is the critical mass for U^{235} fission?

6. What are the string size and dimension of space in string theory?



3. Notations and Useful Approximations

Throughout the course we'll use the following notations:

Symbol	Meaning
$=$	Equality
$\propto$	Proportionality, LHS and RHS could have different units
$\sim$	Same order of magnitude, up to a factor 10
$\simeq$	Approximated equality, up to a factor 2
$\approx$	Approximated equality, up to 30%

Some useful approximated equalities:

$$\sqrt{10} \simeq \pi \simeq 3 \simeq e$$
$$e^3 \approx 20 \ (< 1\% \text{ error})$$
$$\pi^6 \approx e^7 \approx 2^{10} \approx 1000$$
$$\ln 2 \simeq 0.7$$
$$\ln 10 \simeq 2.3$$

4. Dealing with large numbers:

We use scientific notations and retain only the leading one or two digits: more than that is useless for our purposes and may be misleading.

It is good to have some comparative estimates of large numbers. For example, let's find a few estimates of physical quantities that give roughly the same large number, say the number of stars in the sky, or rather in the entire observable Universe.

1. The number of stars in the observable Universe is roughly the number of galaxies, $N_{gal} \sim 10^{11}$, times the number of the stars in a galaxy, $N_{star/gal} \sim 10^{11}$. Hence,
$$N_{stars} \sim N_{gal} N_{star/gal} \sim 10^{22} \, ;$$
2. The number of fine grains of sand (radius ~ 0.05 mm) needed for a single-layer cover of Earth,
$$N \sim \left(\frac{R_{\oplus}}{R_{sand}} \right)^2 \sim 10^{22} \, ;$$
3. And the number of nucleons in an ant, which weighs ~ 0.5 mg,
$$N \sim \frac{M_{ant}}{M_p} \sim 10^{22} \, .$$

Can you find some physically-motivated estimates that give, say, 10^{30}?

B. Estimation Methods

Let's start by importing the classic Fermi problem, "how many piano tuners are in Chicago", and estimate how many car mechanics (people who actually fix cars) work in Be'er-Sheva.

1. *Guesstimate*

The first method to answer such a question is simply to guess - in time, our intuition will become better and better, and our guesses will become closer to the real answer. We call such an answer a guesstimate (guess + estimate), because it's not really a pure guess - we have in mind the important parameters, and we can tell what range of numbers sounds reasonable.

(בעברית, אפשר אולי לתרגם זאת "השערכה".)

In this case, given Be'er Sheva's population of $\sim 200,000$, it's unlikely that the number will be below 50 or above 5000. Any guess in this range will do.

2. Fermi Estimates

A second method is the Fermi estimate (AKA Fermi problem or Fermi question). The idea is to divide the problem into a product of manageable sub-problems, and estimate each of them separately.

In our example, let's estimate how many garage work-hours are needed in Beer sheva per year; knowing how many work days there are in a year would then give us an estimate of the number of mechanics. We estimate that the population in Beer-Sheva is about $N_{BS} \simeq 2 \cdot 10^5$, a family consists on average of about $N_{family} \simeq 4$ members, each family has about $N_{cars/family} \simeq 3/2$ cars, each car visits the garage about $N_{visits/year} \simeq 3$ times a year, mechanics work on each car for a total of about 4 hours, a mechanic's work day is about 9 hours so $t_{work}/t_{day} \sim 4/9$, and there are about $N_{workingdays} \sim 200$ working days in a year. Multiplying these factors, we obtain

$$N_{BS} \times \frac{1}{N_{family}} \times N_{cars/family} \times N_{visits/year} \times \frac{t_{work}}{t_{day}} \times \frac{1}{N_{working\,days}} \sim 500 \text{ mechanics.} \quad (1)$$

Of course, there are many ways to reach such an estimate. For example, one may be familiar with the industrial part of town and estimate that it harbors about $N_{garage} \simeq 100$ garages. This can be multiplied by the typical number of mechanics in a garage, which we may estimate as $N_{mechanic/garage} \simeq 5$. We obtain the same result as before,

$$N_{garage} N_{mechanic/garage} \sim 500 \text{ mechanics.} \quad (2)$$

We got the same number by coincidence, as our estimates are not that accurate. It's fine if our estimates are off by a factor of a few. But if they are not within the same order of magnitude, we better rethink what we've done, and perhaps try a third approach.

Comments:

- Guesstimating is the 'kick-boxing' of physics - there are almost no rules. You may use any information you have available. For example, one may know for some reason that on average, there are 342 cars per 1000 people in Israel, in close agreement with the 375 we find in estimate 1. Similarly, the Yellow Pages list 119 garages in Beer-Sheva, which substantiates estimate 2.

- It is important however to be honest: don't fidget with one estimate in order to match another. It is best to keep your estimates as uncorrelated as you can, as we next show.

- Why do such estimates work at all? We've multiplied several factors, each of them estimated with some considerable uncertainty. Why didn't our mistakes add up to a huge error? Let's assume that our individual errors are uncorrelated (a reasonable assumption if we properly divide the problem). For simplicity, we assume that these errors are all of the same magnitude.

  Denote our estimate $y = x_1 \cdot ... \cdot x_n$, so $\ln y = \sum_{i=1}^{n} \ln x_i$ and its variance is

  $$< (\ln y)^2 >= \sum_{i=1}^{n} < (\ln x_i)^2 >= n < (\ln x_i)^2 > , \tag{3}$$

  so the fractional error we make in y is of order its logarithmic standard deviation,

  $$\sigma(\ln y) = \sqrt{n}\sigma(\ln x) , \tag{4}$$

  growing only as the square root of the number of estimates. For example, in the case of 9 estimated factors, each with a fractional error of order 2, the standard deviation is about $\sim 2^{\sqrt{9}} = 8$ - within an order of magnitude.

- Therefore, when estimating some sub-quantity, a logarithmic interpolation works better than a linear one. For example, if you think there are between 100 and 10,000 beatles in some deserted field, use 1000 rather than the mean 5005.

- Estimating the result in more than one method is essential in order to confirm the result and improve your guesstimates.

- If a quantity seems both overestimated and underestimated, you probably got it right!

Some practice questions

1. How many licks are required to finish a lollipop?

2. How long does your hair grow throughout your life?

3. How many books are in the University's library?

4. How much money is inside an armored money truck?

3. Scaling Arguments

A third method to estimate some value, is to use a similar estimate we already know. For example, if we already figured out how many mechanics work in Be'er Sheva, it would be easy to estimate how many people own a garage in Be'er Sheva, or how many mechanics work in Tel-Aviv.

Let's say we have one estimate A based on parameters $a_1, a_2, ..., a_n$, and we want to estimate another value B from a set of parameters $b_1, b_2, ..., b_n$. We may be able to extrapolate the new result using a scaling of the form

$$B = A \cdot \left(\frac{a_1}{b_1}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{a_n}{b_n}\right)^{\alpha_n} . \tag{5}$$

For example, let's estimate the gravitational acceleration on the surface of the moon. We know it is $g_☽ = GM_☽/R_☽^2$, but this requires us to estimate both the mass and the radius of the moon. Say we only know that the radius of the moon is $R_☽ \simeq R_\oplus/4 \simeq 1700$ km. Defining $\bar{\rho}$ as the average mass density of a body, we may use the scaling

$$g = \frac{GM}{R^2} = \frac{G\bar{\rho}R^3}{R^2} \propto \bar{\rho}R \,, \tag{6}$$

to write

$$g_☽ = g_\oplus \frac{\bar{\rho}_☽}{\bar{\rho}_\oplus}\frac{R_☽}{R_\oplus} . \tag{7}$$

If we assume that the average density of the moon is roughly the same as the Earth's, we obtain $g_☽ \sim \frac{g_\oplus}{4} \approx 2.5$ m/s^2. In comparison, the actual result is about $g_☽ \approx 1.5$ m/s^2, so we're only slightly off.

We can now refine our result. We know that the average mass density of Earth's crust, $\bar{\rho}_{crust} \approx 3$ ton/m^3, is somewhat smaller than the total Earth average, $\bar{\rho}_\oplus \approx 5$ ton/m^3. If we assume that the moon is similar in nature to the Earth crust's, as expected if the moon was created from a piece of Earth's crust broken off by a collision, then we may estimate $\bar{\rho}_☽ \approx \bar{\rho}_{crust}$, and obtain $g_☽ \sim 1.5$ m/s^2. Our estimate thus taught us something new not only about moon walking, but also an important clue to the possible history of the moon!

Comments:

- In this example we obtained $\alpha = 1$, i.e. g was linear in R. Of course, this is not always the case. A good physical understanding is often needed to determine the α's.

For example, in problem set 1 we show that the weight m a man of weight M can lift scales as $m \propto M^{2/3}$.

- It is good practice to divide a problem into parts and use notations, such as $\bar{\rho}$, for each part; sometimes this helps recognize possible scaling arguments. It is instructive to present results in scalable form, such as $g \simeq 10(M/M_\odot)(R/R_\odot)^{-2}$ m s^{-2}, and to examine the limits of such scaling.

4. Dimensional analysis

In many problems we can obtain an approximate solution, up to a numerical factor of order unity, simply by dimensional considerations. Physically, this is based on the fact that Nature does not care which units we use. Formally, we may introduce the Buckingham Π theorem.

Consider a system described by n variables $q_1, q_2, ..., q_n$ of arbitrary units. A physical law can be written as $f(q_1, q_2, ..., q_n) = 0$ for some function f, possibly with many zero curves in $\vec{q}$ space. The Buckingham Π theorem states that the law may be rewritten in the form $F(\Pi_1, \Pi_2, \ldots, \Pi_{n-k}) = 0$, where k is the number of independent dimensions among the q_i, and $\Pi_j = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_n^{\alpha_n}$ are independent dimensionless products of the q_i. The Π variables are called dimensionless groups; their choice is not unique.

Example - Pendulum: We wish to estimate the time period τ of a pendulum with length $l = 1$ m and mass $M = 1$ kg, which is released from an initial angle $\theta_0 = 30°$.
The parameters in the problem are: $\vec{q} = \{\theta_0, m, l, g, \tau\}$, so $n = 5$.
(Don't forget to include here the parameter we are looking for!)
The independent dimensions can be chosen as $\{M, L, T\}$, so $k = 3$.
(Any other set of 3 independent dimensions spanning the parameters would be equally good.)
Hence, we should be able to write $F(\Pi_1, \Pi_2) = 0$ for some choice of Π_1, Π_2.
We notice that m is the only parameter with some units of mass, so no Π parameter can include it. Hence, τ cannot depend on mass!
Now, an evident choice is $\Pi_1 = \theta_0$ and $\Pi_2 = l/(g\tau^2)$.
In matrix form, we may obtain each Π as a non-trivial, independent solution to the equation $Q \cdot \vec{\alpha} = 0$, where element (j, i) of the matrix $Q \in M_{k\times n}$ is the power-law index of dimension

j in the dimensions of parameter q_i. In our case, this equation becomes

$$Q \cdot \vec{\alpha} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = 0\,, \tag{8}$$

and our corresponding choice of dimensionless groups is

$$\Pi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \Pi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -2 \end{pmatrix}. \tag{9}$$

Thus, the answer must have the form $F(\Pi_1, \Pi_2) = F(\theta_0, l/(g\tau^2)) = 0$ for some function F. We may rewrite this as $\Pi_2 = \tilde{F}(\Pi_1)$, where by construction $\tilde{F}(\Pi_1)$ gives the value of Π_2 for which $F(\Pi_1, \Pi_2) = 0$. Note that in general, $\tilde{F}$ could be multi-valued (even if F has a single zero curve) or have an undefined or non-physical value for some Π_1 range.

We conclude that

$$\tau = \tilde{F}(\theta_0)\sqrt{\frac{l}{g}}\,. \tag{10}$$

If the limit $\lim_{\theta_0 \to 0} \tilde{F}(\theta_0) \neq 0$ exists, then in the small θ_0 angle limit we get $\tau = A\sqrt{l/g}$, where A is a dimensionless constant which is usually of order unity in our three-dimensional world, as our physical laws do not have huge constants. (A may substantially differ from unity in some cases, such as in tunneling problems, and if we forgot to include an important dimensionless parameter such as m_p/m_e.) We therefore adopt $A \sim 1$, and estimate that $\tau \sim 0.3$ sec. In practice, $A = 2\pi$, so we are somewhat off, but good to an order of magnitude.

We may have worried that the period τ could also depend on other factors, such as the velocity of the entire pendulum due to Earth's rotation,

$$v_s = \frac{2\pi R_\oplus}{24\text{h}} \sim 400 \text{ m/s}\,. \tag{11}$$

Using the Buckingham Π theorem, we now have $n = 6,\ k = 3,\ n - k = 3$, so in addition to the two Π variables as before, we also need $\Pi_3 = \tau g/v_s$. Here the solution becomes

$$F\left(\theta_0, \frac{l}{g\tau^2}, \frac{\tau g}{v_s}\right) = 0\,. \tag{12}$$

As we saw above, if Π_3 is negligible we get $\Pi_2 \sim 1 \implies \tau \sim 0.3$ sec. If, on the other hand, Π_2 is negligible, we get $\Pi_3 \sim 1 \implies \tau \sim v_s/g \sim 40$ sec. There is a difference of two orders of magnitude between these two results, so only one of them is likely to be relevant. However, our dimensional analysis alone doesn't tell us which one is it. We must therefore supplement our estimate by additional information. For example, we know from experience that the period of such a pendulum is about a second. We conclude that Earth's rotation can be safely neglected in the analysis if we are not interested in $\lesssim 1\%$ accuracy.