Lecture 2a: Scales in Nature

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A. More back-of-the-envelope techniques

In the previous lecture we mentioned several methods for estimating physical quantities to an order of magnitude accuracy:

1. Guesstimates;
2. Fermi Estimates (Divide & Conquer);
3. Scaling Arguments;
4. Dimensional Analysis.

Before outlining scales in physics, we discuss some aspects of back-of-the-envelope estimates.

1. Order-of-magnitude precision is difficult in high-dimension problems

Consider a cube with a side of length d and a ball of diameter d. Their volumes in n dimensions are given by:

$$V_c = d^n \qquad \text{and} \qquad V_b = \frac{(\pi d^2/4)^{n/2}}{\Gamma[1+(n/2)]},$$

where Γ is the Gamma function. The ratio V_c/V_b, as a function of the dimension, is

n	1	2	3	4	5	6	7	8	9	10
V_c/V_b	1	1.3	1.9	3.2	6.1	12.4	27.1	63.1	155	402

This illustrates how rough estimates can be correct up to an order of magnitude in lower dimensional spaces. When the number of dimensions increases, even simple geometric approximations rapidly lower the accuracy.

2. Knowing the Physics behind the scenes

The estimates we discussed, and in particular methods 3 and 4, require an understanding of the underlying physical processes. This improves with study and with practice. Naturally, your estimates will often fail because the physics underlying a problem is different than you thought. You should be doubly pleased when this happens: (i) you get to learn new physics; and (ii) aren't you happy you found your misconception after only a back-of-the-envelope estimate, rather than after a long derivation or a numerical computation?

As a first step, one should identify the applicable physical regime, the relevant physical scales, the laws governing the behavior of the system, and important vs. negligible aspects of these laws for the problem at hand. Much of the following lectures is meant to expand you knowledge of various physical regimes and to provide you with a set of tools appropriate to tackle a wide range of problems.

3. *Clever parameterization*

It is often useful to express physical quantities as a function of relevant scales or in comparison to other objects you are familiar with. For example, when discussing celestial objects such as planets or moons, it's useful to express their mass and radius in relation to those of Earth, which you know well. Thus, instead of simply giving the numerical result $g = GM/R^2 \approx 10$ m s^{-2}, it is better to write

$$g = \frac{GM}{R^2} \approx 10 \left(\frac{M}{M_{earth}}\right)\left(\frac{R}{R_{earth}}\right)^{-2} \text{ m s}^{-2}.$$

This immediately tells you what is the gravitational acceleration on the moon, if you can estimate its mass and size in comparison to Earth. Of course, it may be better to change the parameterization if you are dealing with different objects. For example, use the dimensions of the sun when studying stars.

4. *Finding an approximate solution to an equation without solving it*

Next, consider the nature of physical law, typically encapsulated in some equation. Most of our back-of-the-envelope estimates are equivalent to finding an approximate solution to such an equation without actually solving it. One can often obtain reasonable such estimates, by identifying the relevant scales in the problem, neglecting processes operating on irrelevant scales, approximation differentials, integrals, etc., and ending up with a simple, typically algebraic, formula.

In any equation you meet, find the minimal set of physical scales involved, identify the relevant dimensionless physical quantities (for example m_e/m_p, the Reynolds number, etc. - if there are any), and figure out the roles of these scales and numbers. For example, consider the Schrödinger's equation for a free particle:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi,$$

which may be written in terms of a characteristic length, which turns out to be Compton's (reduced) wavelength $\lambda \equiv \hbar/mc$, and the timelike coordinate $x_0 = ct$,

$$i\partial_{x_0}\psi = -\frac{\lambda}{2}\nabla^2\psi.$$

As we later show, λ is the typical scale below which new particles are naturally generated. (Have you realized this when previously dealing with Schrödinger's equation?)

5. Know your equations...

As an example of the need to know the underlying physics, consider the evolution of a particle with some spin s. Do you know which equation determines its evolution? Are you familiar with the following equations:

spin	equation name	equation
$s = 0$	Klein-Gordon	$-\Box\psi \equiv -\frac{1}{c^2}\partial_{t,t}\psi + \nabla^2\psi = \lambda^{-2}\psi$
$s = 1/2$	Dirac	$i\gamma^\mu\partial_\mu\psi = \lambda^{-1}\psi$
$s = 1$	Proca	
$s = 3/2$	Rarita-Schwinger	
any s	Bargmann-Wigner	

Don't worry if you never heard of some of them. For most of our estimates, we will approximate each component of the wavefunction as evolving according to the Klein-Gordon equation. But appreciate that in some cases this may not suffice, and you may miss interesting effects if you don't know, say, the Proca equation.

By the way, did you digest our previous comment and notice that in all these equations, the only length scale is Compton's wavelength, λ?

6. *Quick fixes for equations*

Solving differential equations can be difficult. Integro-differential equations are even harder. So it's useful to find shortcuts for getting quick results.

As a rule of thumb, one can often replace derivatives by fractions,

$$\frac{d\Psi}{dL} \;\rightarrow\; \pm\frac{\Psi}{L}.$$

The result is correct dimensionally, and often gives reasonable order of magnitude estimates if you know to choose the correct plus or minus sign. Such a substitution fails when the logarithmic derivative α, defined by

$$A \sim B^{\alpha} \quad \Leftrightarrow \quad \frac{dA}{dB} \sim \alpha\frac{A}{B},$$

is very large or very small. In fact, α quantifies the error in the above substitution.

Similarly, when a function behaves nicely in a certain parameter range, a quick estimate of an integral would be

$$\int_{x_i}^{x_f} f(x)dx \sim \langle f(x)\rangle (x_f - x_i),$$

where $\langle f(x)\rangle$ if a typical average value in the integration range.

B. Scales in Nature

We next review various scales of nature. Starting with some physical constants, we briefly work our way to scales in our lives, and in the Universe.

1. *Physical Constants*

The physical constant you need to remember depend on your discipline. It is generally advised to memorize at least the general constants

$c \simeq 3.0 \times 10^{8}$ m s^{-1}

$\hbar \simeq 1.1 \times 10^{-34}$ g m^{2} s^{-1}

$G \simeq 6.7 \times 10^{-11}$ m^{3} kg^{-1} s^{-2}

$m_p \simeq 2000 m_e \simeq 1.6 \times 10^{-27}$ kg

$$\alpha = \begin{cases} \frac{e^2}{(4\pi\epsilon_0)\hbar c}, & \text{(MKS)} \\ \frac{e^2}{\hbar c}, & \text{(CGS)} \end{cases} \simeq \frac{1}{137}$$

Here, α is the fine structure constant for the electromagnetic force. We later denote it as α_E, to distinguish it from the other interactions, which we denote α_G (gravitation), α_W (weak force), and α_S (strong force). Can you estimate these coupling coefficients?

2. Units

We will use interchangeably SI and Gaussian (CGS) units, as customary in the specific physical field at hand. In addition, many disciplines have their own set of units. For example, an astrophysicist should memorize

1 eV $\simeq 1.6 \times 10^{-19}$ J

1 year $\simeq \pi \times 10^7$ s

1 pc $\simeq 3.1 \times 10^{16}$ m

etc.

3. Daily scale estimation

Have you estimated the distance from your house to your car/bus stop/pub? How much your backpack weighs? How long it takes a raindrop to fall from a cloud? As we already mentioned, the more you estimate things in your daily life, the better and faster you'll become in such estimates, and more interesting questions will surface.

People typically grasp and can intuitively estimate time periods ranging from a fraction of a second to decades, as we have experience on these scales. Our concept of time begins to fail when dealing with timescales shorter than a milliseconds or longer than centuries. Time measurements, once done with crude technique's such as counting heartbeats, are now automatically available as most people carry watches.

Similarly, people can estimate masses ranging from grams to hundreds of killos. However, one is easily fooled when estimating mass (or equivalently, force) because we usually estimate the weight of an object judging by the strength needed to lift it. Alas, we actually feel the moment of the force, and so our estimate will depend on the shape and size of the object, and on how we hold it. For example, trying to lift a weight farther from our body will require a larger moment, giving the illusion that the object is heavier.

Similar arguments apply to length scales. Many people are accustomed to making accurate length measurements using their shoe size, the distance between their thumb and finger when maximally stretched out (~ 25 cm), counting footsteps, etc.

It is sometimes useful to measure the angular extent of an object. By holding out your thumb as far as possible, you always have your own goniometer with you. A stretched out thumb is $\sim 2^\circ$ wide, and a pinky is $\sim 1^\circ$ - measure and memorize the dimension of your own hand and fingers. Now you can measure the angular extent of, say, the moon. Can you guess this angle? For example - do you think you can cover the entire moon with your stretched out thumb? (Answer: the moon spans about $0^\circ.5$, and therefore can be covered by either stretched out thumb or pinky.)

The subtended angle θ provides a useful and easily available tool for measuring the size L of a distance object (say, the moon), provided that you can estimate the distance d between you and the object. When the angle is small, the approximation $L \simeq \theta d$ (with an error $\tan(\theta/2)/(\theta/2)$) is good to better than 30% even when $\theta = 45^\circ$.

If an object is moving at a nearly constant velocity, angular distances can also be used to measure time. For example, the sun takes 24 hours to travel it's 360° across the sky, so the angular velocity is

$$\frac{360^\circ}{24 \text{ hr}} = \frac{15^\circ}{1 \text{ hr}} = \frac{1^\circ}{4 \text{ min}}.$$

Thus, the time it takes the sun to traverse 1° in the sky is roughly 4 minutes. You can use this to estimate how much time remains until sunset.

4. Length and mass scales in the Universe

Next, we zoom out and inspect all length and mass scales in the Universe. The physics we wish to discuss spans some ~ 60 orders of magnitude in length, and some ~ 90 orders of magnitude in mass. There are good video clips and animations that traverse these many scales; see for example http://htwins.net/scale2/ . In the following figures we use a logarithmic plot to illustrate the mass–length scales phase space. You can do the same for other phase spaces, such as length–time, mass-velocity, etc.

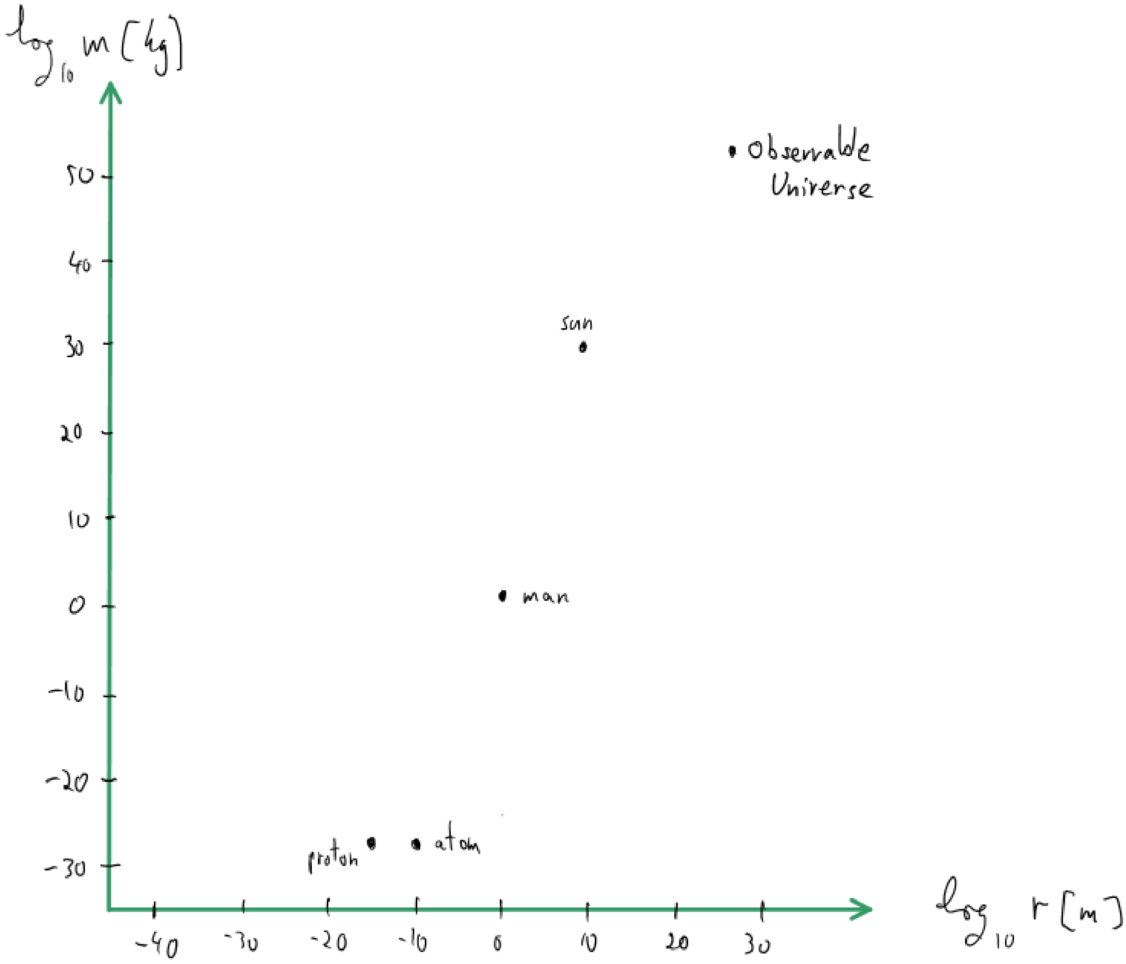

FIG. 1: Illustration of the $\log(r) - \log(m)$ phase space, with labeled points representing some physical objects.

This already shows some interesting patterns. Notice for example the similar mass but the different size of a proton and an atom. Also notice that the atom, the man, and the sun seem to lie on one line. What could be the reason for that? Can you add more objects to this plot?

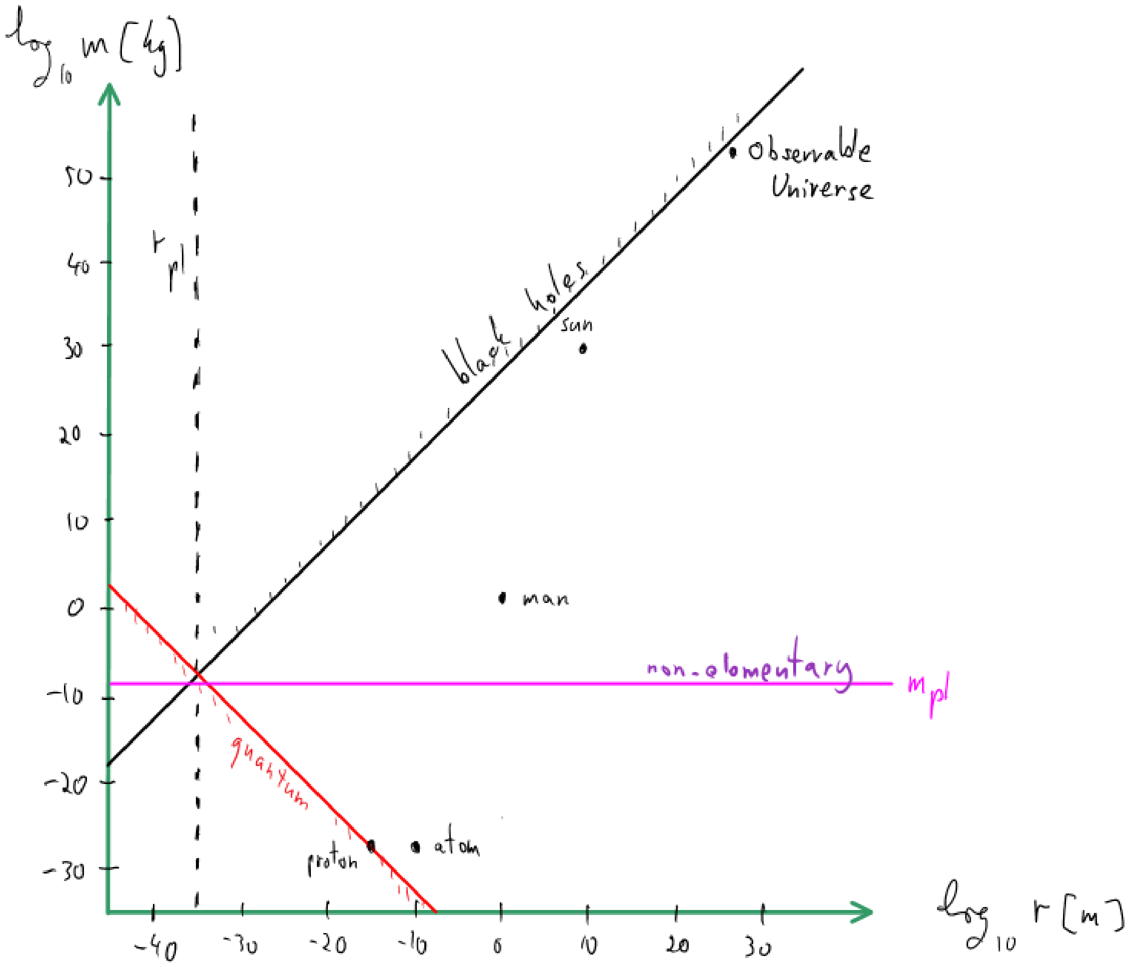

FIG. 2: Illustration of some different physical regimes.

The black-hole line is approximated by the Schwarzschild radius $r_s = 2Gm/c^2 \Rightarrow \log(m) \propto log(r)$. Notice how close our Universe is to being a black hole – what does that mean?

The transition to quantum behavior is typically associated with the deBroglie wavelength, $\lambda_{dB} = \lambda/(\gamma\beta)$. Instead, we plot the Compton wavelength line, $r = \lambda$, which demarks the transition to the quantum field theory regime. Notice that the atom lies above this scale, but the proton does not - what does this mean?

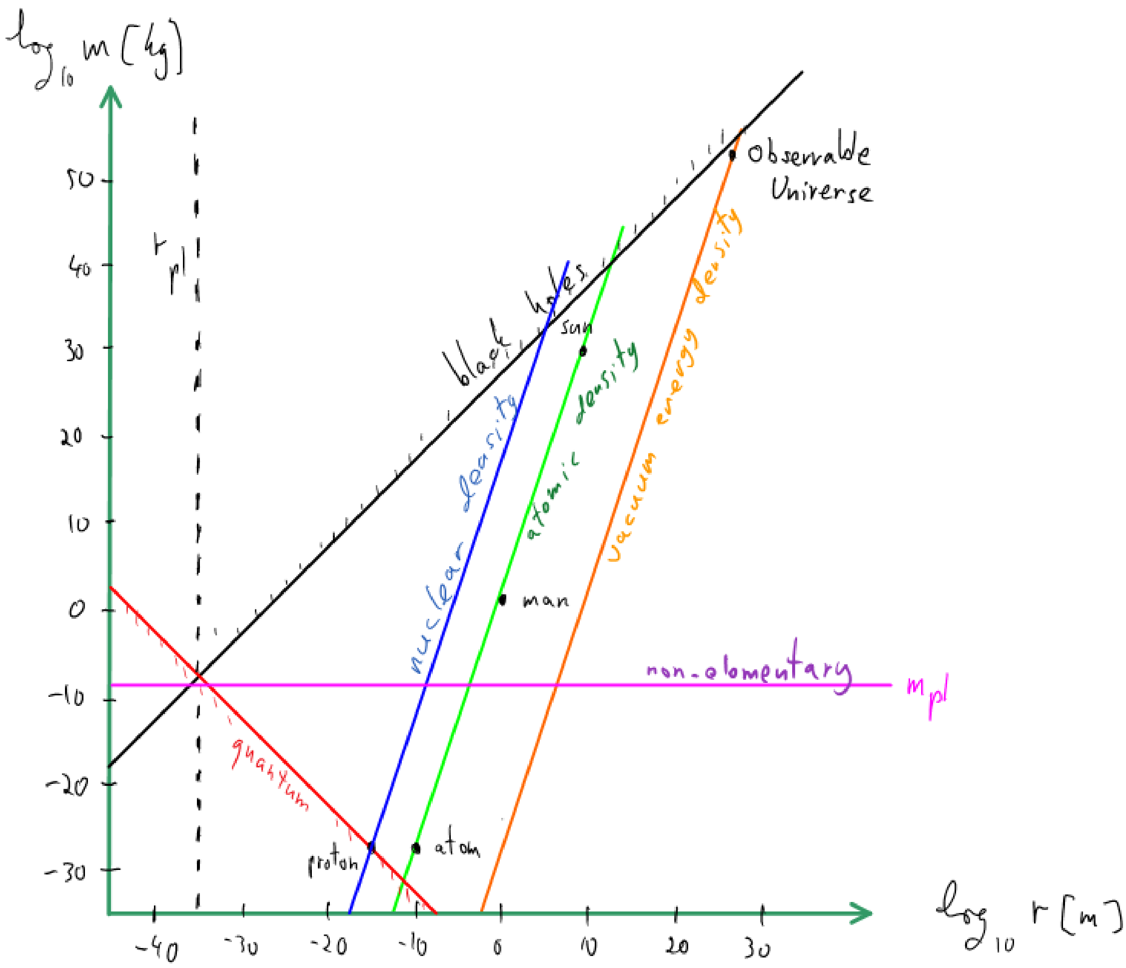

FIG. 3: Illustration of some interesting mass densities $\rho \simeq m/r^3 \Rightarrow \log(m) \propto 3\log(r)$.

Notice that the proton has about the same mass density as that of any nucleus. What does that tell you about the structure of the nucleus and its underlying forces?

You may gain deeper insights by adding more lines and objects to this (and similar) plot.