HW 8

1 Potential Energy

Force is given by $F_\rho = \frac{a}{\rho^2}$, $F_\varphi = \frac{b\sin\varphi}{\rho^2}$ (cylindrical coordinates).
Is the force conservative?
*Note that in cylindrical coordinates the rotor is given by:

$$\overrightarrow{\nabla} \times \overrightarrow{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix}$$

If yes, find the potential energy.
What is conserved in this force?

Solution:

The force is conservative if

$$\overrightarrow{\nabla} \times \overrightarrow{F} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial_\rho & \partial_\varphi & \partial_z \\ \frac{a}{\rho^2} & \rho\left(\frac{b\sin\varphi}{\rho^2}\right) & 0 \end{vmatrix} =$$

$$= \frac{\left[-\partial_z\left(\rho\left(\frac{b\sin\varphi}{\rho^2}\right)\right)\right]\hat{\rho} + \left[\partial_z\left(\frac{a}{\rho^2}\right)\right]\hat{\varphi} + \left[\partial_\rho\left(\rho\left(\frac{b\sin\varphi}{\rho^2}\right)\right) - \partial_\varphi\left(\frac{a}{\rho^2}\right)\right]\hat{z}}{\rho} \Rightarrow$$

where $\overrightarrow{F}$ is not dependent on z and F_ρ is not dependent on φ, so

$$\Rightarrow \frac{\partial_\rho\left(\frac{b\sin\varphi}{\rho}\right)}{\rho}\hat{z} = \frac{-\frac{b\sin\varphi}{\rho^2}}{\rho}\hat{z} \Rightarrow \overrightarrow{\nabla} \times \overrightarrow{F} = -\frac{b\sin\varphi}{\rho^3} \neq 0$$

The force is not conservative for $b \neq 0$.
We could equally check if mixed partial are equal:

$$\frac{1}{\rho}\frac{\partial F_\rho}{\partial\varphi} = -\frac{1}{\rho}\frac{\partial^2 V}{\partial\varphi\partial\rho} = 0$$

$$\frac{\partial F_\varphi}{\partial\rho} = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial V}{\partial\varphi}\right) = \frac{b\sin\varphi}{\rho^2}$$

$$\frac{\partial F_\varphi}{\partial\rho} \neq \frac{1}{\rho}\frac{\partial F_\rho}{\partial\varphi}$$

Which means that there is no potential function V for $\overrightarrow{F}$.

2 Conservative Force

For the given force

$$\boldsymbol{F} = -k\left(x-y\right)^2\hat{\boldsymbol{x}} + k\left(x-y\right)^2\hat{\boldsymbol{y}} + Kz\hat{\boldsymbol{z}},$$

1. Is the force a conservative force? If it is not, demonstrate it. If it is, find its potential.
2. A body moves from position $(0,0,0)$ in a straight line to position $(D,D,0)$, and then to (D,D,D). Calculate the work done by the force, using path integral.

Solution:

1. Let us calculate the rotor of $\boldsymbol{F}$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= \left(\partial_y F_z - \partial_z F_y\right)\hat{\boldsymbol{x}} + \left(\partial_z F_x - \partial_x F_z\right)\hat{\boldsymbol{y}} + \left(\partial_x F_y - \partial_y F_x\right)\hat{\boldsymbol{z}} \\ &= (0-0)\,\hat{\boldsymbol{x}} + (0-0)\,\hat{\boldsymbol{y}} + \left[2k\left(x-y\right) - 2k\left(x-y\right)\right]\hat{\boldsymbol{z}} \\ &= 0.\end{aligned}$$

The force is conservative, let us find its potential U, which must follow $\boldsymbol{F} = -\boldsymbol{\nabla}U$

$$\begin{aligned}U &= -\int F_x dx = \frac{k}{3}\left(x-y\right)^3 + C_x\left(y,z\right), \\ U &= -\int F_y dy = \frac{k}{3}\left(x-y\right)^3 + C_y\left(x,z\right), \\ U &= -\int F_z dz = -\frac{K}{2}z^2 + C_z\left(x,y\right).\end{aligned}$$

Thus

$$\frac{k}{3}\left(x-y\right)^3 + C_x\left(y,z\right) = \frac{k}{3}\left(x-y\right)^3 + C_y\left(x,z\right) = -\frac{K}{2}z^2 + C_z\left(x,y\right),$$

from the left equation we conclude $C_x = C_y = C\left(z\right)$, then from the right equation we find

$$\begin{aligned}C\left(z\right) &= -\frac{K}{2}z^2 + C_0 \\ C_z\left(x,y\right) &= \frac{k}{3}\left(x-y\right)^3 + C_0.\end{aligned}$$

Plugging all into U we find

$$U = \frac{k}{3}\left(x-y\right)^3 - \frac{K}{2}z^2 + \cancel{C_0}.$$

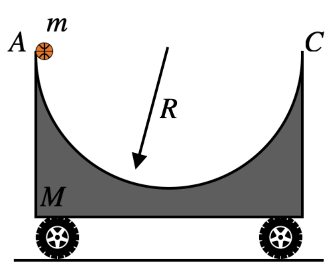
2. The path integral is

$$\begin{aligned}
W &= \int \boldsymbol{F}\cdot d\boldsymbol{r} \\
&= \int_{(0,0,0)}^{(D,D,0)} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{(D,D,0)}^{(D,D,D)} \boldsymbol{F}\cdot d\boldsymbol{r} \\
&= \int_{(0,0)}^{(D,D)} \left[-k\left(x-y\right)^2\hat{\boldsymbol{x}} + k\left(x-y\right)^2\hat{\boldsymbol{y}}\right]\cdot\left[d\boldsymbol{x}+d\boldsymbol{y}\right] - \int_0^D Kz\hat{\boldsymbol{z}}\cdot d\boldsymbol{z} \\
&= \int_0^D -k\left(x-y\left(x\right)\right)^2 dx + \int_0^D k\left(x\left(y\right)-y\right)^2 dy - \int_0^D Kz\,dz \\
&= \int_0^D -k\cancel{(x-x)}^2 dx + \int_0^D k\cancel{(y-y)}^2 dy + \frac{1}{2}KD^2 \\
&= \frac{1}{2}KD^2,
\end{aligned}$$

where the first integral is along the line $y=x$, and since $x-y=0$ it vanishes along this path.

3 Sliding Cart

A cart with mass M stationed on a frictionless horizontal plane. On the cart installed a frictionless surface in the shape of half a circle with radius R (see figure). A ball with mass m is allowed to slide from point A (the top of the plane).



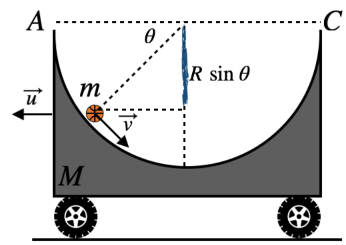
1. Write down the equations for momentum and energy conservation. Denote the ball's velocity components (relative to stationary frame) by v_x and v_y, and those of the cart by u_x and u_y.
2. Does the ball reach the other side of the surface (point C)? If so, find its velocity and the position of the cart (relative to stationary frame) at the moment the ball reaches point C.

Solution:

1. Let us define the point of reference for the potential energy to be at the height of point A (or C equivalently), thus at any point

$$E_p + E_k = -mgR\sin\theta + \frac{1}{2}m\left(v_x^2+v_y^2\right) + \frac{1}{2}Mu_x^2 = \text{Constant},$$

where we defined θ as shown in the figure



The velocity of the ball, relative to the cart, is

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u} = \boldsymbol{v} - u_x \hat{\boldsymbol{x}},$$

therefore

$$\sin\theta = \frac{|v_x'|}{|\boldsymbol{v}'|} = \frac{|v_x - u_x|}{|\boldsymbol{v} - u_x\hat{\boldsymbol{x}}|} = \frac{|v_x - u_x|}{\sqrt{(v_x - u_x)^2 + v_y^2}}.$$

Plugging that into the energy conservation equation we find

$$-mgR\frac{|v_x - u_x|}{\sqrt{(v_x - u_x)^2 + v_y^2}} + \frac{1}{2}m\left(v_x^2 + v_y^2\right) + \frac{1}{2}Mu_x^2 = 0.$$

As for the momentum conservation, the momentum is only conserved in the x direction, in which

$$mv_x + Mu_x = 0.$$

2. At point C the ball would have $v_y = 0$, and since the ball does not leave the surface, we must also require that at this point $v_x = u_x$. Looking at the momentum conservation we find that the only configuration for this to happen is when

$$u_x = v_x = 0,$$

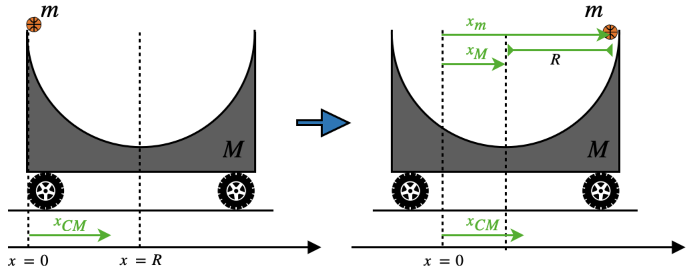
that means that the ball reaches point C where its velocity becomes 0. (we could also see that from the expression for $\sin\theta$ at $\theta = \pi$)
Since no external forces act on the x direction, we can expect the center of mass to remain constant, defining the origin at point A, we find the initial position of the CM to be

$$x_{CM}(t_A) = \frac{mx_m(t_A) + Mx_M(t_A)}{m+M} = \frac{0 + MR}{m+M} = \frac{MR}{m+M}.$$

Therefore, when the ball reaches point C we find

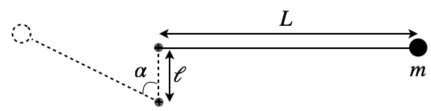
$$x_{CM}(t_C) = \frac{mx_m(t_C) + Mx_M(t_C)}{m+M} = \frac{m(x_M(t_C) + R) + Mx_M(t_C)}{m+M} = \frac{MR}{m+M} \quad \to \quad x_M(t_C) = R\frac{M-m}{m+M},$$

implying that it really did move left.



4 Yo-Yo

A yo-yo with mass m is connected to a nail via string with length L. Underneath the nail, at distance ℓ, another nail is placed (see figure). The yo-yo is released from rest, and when it reaches the second nail it continues motion as the string remains stretched between the two nails.



1. What is the velocity of the yo-yo when the angle between the two segments of the string reaches α (see figure)?
2. Find α for which the string becomes loose.

Solution:

1. Since the tension acts perpendicular to the path of the yo-yo, and the only other force is gravity, which is conservative, then the energy is conserved during the motion. Choosing the point of reference for the potential energy to be at the bottom nail we can write

$$E_i = mg\ell$$
$$E_f = \frac{1}{2}mv_\alpha^2 + mg\,(L-\ell)\cos\alpha,$$

which gives us

$$v_\alpha = \sqrt{2g\,[\ell\,(1+\cos\alpha) - L\cos\alpha]}.$$

2. The yo-yo undergoes circular motion around the bottom nail up to the point when the string becomes loose. Therefore we may write the force equation as

$$-T - mg\cos\alpha = -m\frac{v^2}{L-\ell} = -m\frac{2g}{L-\ell}\,[\ell\,(1+\cos\alpha) - L\cos\alpha]\,.$$

At the point of loosening the tension vanishes, therefore

$$\cos\alpha = \frac{2}{L-\ell}\,[\ell + (\ell - L)\cos\alpha] \quad \rightarrow \quad \cos\alpha = \frac{2}{3}\frac{\ell}{L-\ell}.$$

We see that when $\ell \to L$ the solution breaks down since the yo-yo does not reach the bottom nail, and also there is a minimal ℓ when we require $\cos\alpha \leq 1$, we find $\ell \geq \frac{3}{5}L$, over which the yo-yo will complete a loop (i.e. the string will never be loose).

5 Central potential and Ionization

A particle is connected to a point (The source of some force that the particle feels) by a central potential

$$u(r) = \frac{A}{r^2} - \frac{B}{r}$$

where r is the distance of the particle from the source of the force and A and B are positive constants.
* The motion of the particle in the radial direction only and therefore can be described as one-dimensional.

1. Find the distance of equilibrium r_0 and show that it is stable.

2. Calculate the energy of ionization ϵ_0 i.e. the amount of work needed in order to move the particle from $r_0$to infinity.

3. Express $u(r)$ in terms of r_0 and ϵ_0 alone.

4. What is the work done by the force derived from $u(r)$ when the particle moving from $r_1 = \left(\sqrt{2}r_0, \sqrt{2}r_0\right)$ to $r_2 = \left(\frac{r_0}{\sqrt{2}}, \frac{r_0}{\sqrt{2}}\right)$ in the $x - y$ plane? Use ϵ_0 to express your answer.

5. It is given that the total energy of the particle is $E = -\frac{3\epsilon_0}{4}$ and that its motion is only radial. Find the values of r where the particle's velocity vanishes.

Solution:

1. Differentiating $u(r)$

$$\frac{du}{dr} = -2\frac{A}{r^3} + \frac{B}{r^2}$$

$\frac{du}{dr} = 0$ for the distance of equilibrium r_0

$$2\frac{A}{r_0^3} = \frac{B}{r_0^2}$$

$$r_0 = \frac{2A}{B}.$$

If $r_0$is a stable equilibrium $\left(\frac{d^2u}{dr^2}\right)_{r_0} > 0$

$$\left(\frac{d^2u}{dr^2}\right)_{r_0} = \left(6\frac{A}{r^4} - 2\frac{B}{r^3}\right)_{r_0} =$$
$$= 6A\left(\frac{B}{2A}\right)^4 - 2B\left(\frac{B}{2A}\right)^3 =$$
$$= \frac{3}{8}\frac{B^4}{A^3} - \frac{2}{8}\frac{B^4}{A^3} > 0$$

2. Using the relation between potential energy and work by its force:

$$\epsilon_0 \equiv W(r_0 \to \infty) = \Delta u = u(\infty) - u(r_0) = -u(r_0) = \frac{B^2}{4A}$$

*See the note at the end of this solution.

3. $u(r) = \epsilon_0\left[\left(\frac{r_0}{r}\right)^2 - 2\left(\frac{r_0}{r}\right)\right]$

4. We note that this force is conservative (it has a given potential) therefore the trajectory does not important - only the initial and final points:

$$W = u(r_2) - u(r_1) =$$
$$= u\left(\sqrt{\frac{r_0^2}{2} + \frac{r_0^2}{2}}\right) - u\left(\sqrt{2r_0^2 + 2r_0^2}\right) = u(r_0) - u(2r_0) =$$
$$= -\epsilon_0 - \left(\epsilon_0\left[\left(\frac{r_0}{2r_0}\right)^2 - 2\left(\frac{r_0}{2r_0}\right)\right]\right) = -\epsilon_0 + \frac{3}{4}\epsilon_0 =$$
$$= -\frac{\epsilon_0}{4}$$

*See the note at the end of this solution.

5. The total energy of the particle is given by

$$E_{tot} = K + u(r)$$

When the velocity of the particle vanishes its kinetic energy also vanishes.

$$\left(\frac{r_0}{r}\right)^2 - 2\left(\frac{r_0}{r}\right) = -\frac{3}{4}$$

$$r^2 - \left(\frac{8r_0}{3}\right) r + \frac{4}{3} r_0^2 = 0$$

$$r_{1/2} = \frac{1}{2}\left[\frac{8r_0}{3} \pm \sqrt{\left(\frac{8r_0}{3}\right)^2 - 4\frac{4}{3}r_0^2}\right] = \frac{4}{3}r_0\left[1 \pm \sqrt{1-\frac{6}{8}}\right]$$

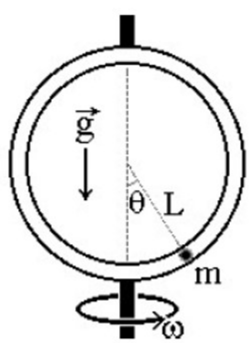
$$r_{1/2} = \frac{4}{3}r_0\left(1 \pm \frac{1}{2}\right)$$

***Note:** In section 2 and section 4 the particle moved in the same direction but we got opposite signs for the work.
In section 2 "we" needed to exert force on the particle **against the potential** in order to move it and that is why the work is positive.
In section 4 the potential did the work on the particle and that is why we got a negative work.

6 Spherical pendulum - Bonus

A bead of mass m is inside a hollowed hoop of radius L which is turning with an angular velocity ω around its axis (vertical relative to ground).

1. Write an expression for the total potential energy $V(\theta)$ of the bead in the frame rotating with the hoop.
2. Use the expression for $V(\theta)$ to find the angle where the bead is in equilibrium and check if they are stable.
3. Sketch $V(\theta)$.



Solution:

1. First let us recognize the forces in the rotating frame of reference:
   Gravitational force $F_g = mg\,(-\hat{y})$- which we know it is conservative.
   Centrifugal force $F_c = m\omega^2\rho(\theta)\hat{\rho}$ - A force in the $\hat{\rho}$ direction which only depend on ρ - its mixed partial must vanish - also a conservative force.
   The centrifugal potential

$$V_c = -\int m\omega^2\rho d\rho = -\frac{1}{2}m\omega^2\rho^2 = -\frac{1}{2}m\omega^2L^2\sin^2\theta.$$

   The gravitational potential

$$V_g = mgL\,(1-\cos\theta)\,.$$

   The total potential energy

$$V(\theta) = mgL\left(1-\cos\theta-\frac{1}{2}\frac{\omega^2L}{g}\sin^2\theta\right).$$

2. The equilibrium point satisfies $\frac{dV}{d\theta} = 0$

$$mgL\left(\sin\theta-\frac{\omega^2L}{g}\sin\theta\cos\theta\right)$$

$$\sin\theta = 0 \ \text{ or } \ \cos\theta = \frac{g}{\omega^2L}$$

$$\theta_1 = 0,\pi \ \text{ or } \ \theta_2 = \pm\arccos\frac{g}{\omega^2L}$$

   Differentiating again in order to find the stable points

$$\begin{aligned}\frac{d^2V}{d\theta^2} &= \frac{d}{d\theta}\left(mgL\sin\theta\left(1-\frac{\omega^2L}{g}\cos\theta\right)\right) = \\ &= mgL\left[\cos\theta\left(1-\frac{\omega^2L}{g}\cos\theta\right)+\frac{\omega^2L}{g}\sin^2\theta\right]\end{aligned}$$

   Define $\omega_c \equiv \sqrt{\frac{g}{l}}$

$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=0} = mgL\left(1-\frac{\omega^2}{\omega_c^2}\right)$$

   For $\omega < \omega_c$ $\theta = 0$ is minimum and is stable. For $\omega < \omega_c$ $\theta = 0$ is unstable.

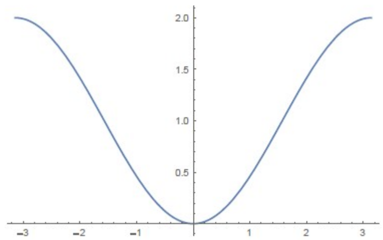
$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\pi} = -mgL\left(1+\frac{\omega^2}{\omega_c^2}\right)$$

   therefore $\theta = \pi$ is unstable for any value of ω.
   Note that for $\omega < \omega_c$ the points θ_2 are irrelevant.
   Assuming $\omega > \omega_c$:

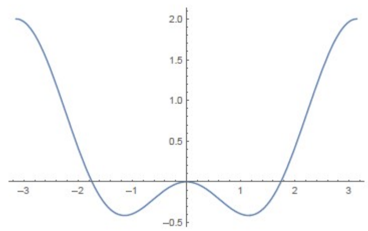
$$\begin{aligned}\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\pm\arccos\frac{g}{\omega^2L}} &= mgL\left(\frac{g}{\omega^2L}(1-1)+\frac{\omega^2L}{g}\left(1-\left(\frac{g}{\omega^2L}\right)^2\right)\right) = \\ &= mgL\left(\left(\frac{\omega}{\omega_c}\right)^2+\left(\frac{\omega_c}{\omega}\right)^2\right) > 0\end{aligned}$$

   and θ_2 are stable

3. The graph for $\omega < \omega_c$

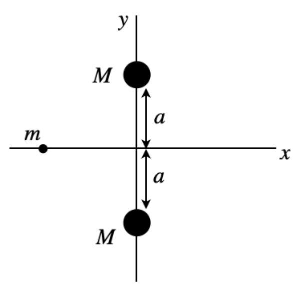


For $\omega > \omega_c$



7 Oscillations of bead with gravitating masses - Bonus

A bead of mass m slides without friction on a smooth rod along the x axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0$, $y = \pm a$ as shown in the figure, and attract the bead gravitationally.



Using the potential energy:

1. Find the point of equilibrium.
2. Find the frequency of small oscillations of the bead about the origin.
3. Draw the potential as a function of x.
4. For what range of energies do we find bounded trajectories?
5. Given energy $E < 0$, what would be the turning points of the trajectory?

Solution:

1. The point of equilibrium can be found directly from the potential which is

$$U = -2\frac{GmM}{\sqrt{a^2 + x^2}},$$

simply by finding its minima . Taking the derivative and equating it to zero yields

$$\frac{dU}{dx} = 2\frac{GmMx_0}{\left(a^2 + x_0^2\right)^{3/2}} = 0 \quad \rightarrow \quad x_0 = 0.$$

2. The frequency of the oscillations may be extracted from the potential as well. Expanding the potential as a Taylor series around the point of equilibrium

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x_0} (x - x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} (x - x_0)^2 + \ldots$$

we find that the first derivative vanishes (as we required to be at x_0), while the second derivative reads

$$\frac{d^2U}{dx^2} = 2GmM\frac{a^2 - 2x^2}{\left(a^2 + x^2\right)^{5/2}},$$

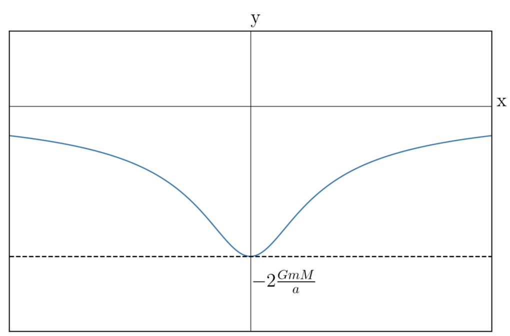
thus

$$U(x) = -2\frac{GmM}{\sqrt{a^2 + 0}} + GmM\frac{a^2 - 0}{\left(a^2 + 0\right)^{5/2}}(x - 0)^2 + \cdots \approx -2\frac{GmM}{a} + \frac{1}{2}\left(\frac{GmM}{a^3}\right)x^2.$$

The first term is the minimal energy of the system, which is constant and does not affect physical quantities, while the second term is the leading term which exhibits the dynamics of the system. The latter resembles the potential energy of a spring, which is $\frac{1}{2}kx^2$, for which we already know the frequency of the oscillations to be $\omega = \sqrt{k/m}$, only with $k = GmM/a^3$. Therefore we can expect the frequency of the oscillations to be

$$\omega = \sqrt{\frac{GM}{a^3}}.$$

3. The potential is symmetric around the y axis, goes to zero for $x \rightarrow \pm\infty$ and has a minimum at $x = 0$ with the value $-2GmM/a$, thus



4. In order for the trajectory to be bounded we require that the trajectory will have turning points in which the kinetic energy vanishes. Due to energy conservation it is easy to see that

$$E = E_k + U \overset{E_k=0}{\rightarrow} E = U(x_{\max}) < 0 \quad \rightarrow \quad E < 0,$$

the energy must be negative, or in equivalently $|E_k(x)| < |U(x)|$. In other words we could say that the energy must be inside the potential well we drew in (3).

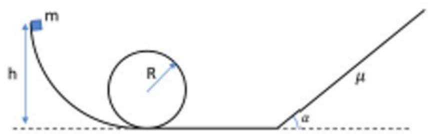
5. Given $E < 0$ we write again energy conservation and plug $E_k = 0$, which yields

$$-|E| = U(x_{\max}) = -2\frac{GmM}{\sqrt{a^2 + x_{\max}^2}} \quad \rightarrow \quad x_{\max} = \sqrt{\left(\frac{2GmM}{|E|}\right)^2 - a^2},$$

which exhibits the property of increasing $x_{\max}$ as we increase the energy ($E < 0$ thus reducing $|E|$), while reducing the energy (increasing $|E|$) is allowed only to a minimal value of $-2gmM/a$ which is the minimal possible energy of the system.

8 Loop of death - Bonus

A track consists of a smooth track, a loop of death of radius R, and a long rough slope with a friction coefficient μ and an angle α (as shown in the figure).
From what minimal height h we need to release the body in the left track in order to it pass the loop of death twice?

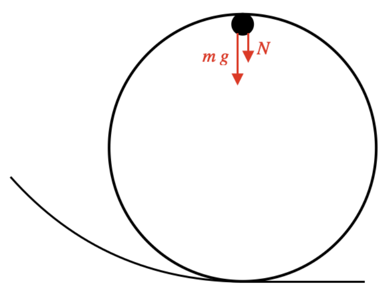


Solution:

Let's write the total energy of the body

$$E = K + V = \frac{1}{2}mv^2 + mgh$$

Starting from the end, the body will preform a circular motion on the loop of death for the second time if at the upper point of the loop the body will still touch it. Meaning that $N > 0$.



Writing the force equation:

$$N + mg = m\frac{v^2}{R}$$

$$0 < N = m\left(\frac{v^2}{R} - g\right)$$

$$v_{min}^2 = gR$$

Therefore

$$(E_2)_{MIN} = \frac{1}{2}m\left(gR\right) + mg2R = \frac{5}{2}mgR$$

where E_2 is the total energy of the body on the top point of the loop of death when it passes it for the second time.
Denote E_1 for the total energy of the body when it stops at some height marked as H in the rough slope. The difference between those two is equal to the work done by friction.
The normal force when the body is on the rough slope

$$N = mg\cos\alpha$$

therefore the friction force is equal to $|f| = \mu mg\cos\alpha$ and directed up the slope.
Calculate the friction work as the body slides down a distance d

$$H = d\sin\alpha$$

and the friction is constant along the slide

$$W_f = -\mu mgd \cos\alpha.$$

Now we can get the minimal value for E_1:

$$(E_2)_{MIN} = (E_1)_{MIN} + W_f$$

$$(E_1)_{MIN} = \frac{5}{2}mgR + \mu mgd \cos\alpha.$$

We can find d by writting explicitly $(E_1)_{MIN}$:

$$(E_1)_{MIN} = mgH = mgd \sin\alpha$$

and

$$\frac{5}{2}R = d\left(\sin\alpha - \mu\cos\alpha\right).$$

Denote E_0 for the total energy of the body when it first released at the left track.
$E_1 - E_0$ equal to the work done by friction when the body slided up the slope. Note that it is the same trajectory as before in the opposite direction, with a friction force acting to an opposite direction as well. Therefore

$$mgh = (E_0)_{MIN} = \frac{5}{2}mgR + 2\mu mgd \cos\alpha.$$

$$h = \frac{5}{2}R\left(1 + \frac{2\mu\cos\alpha}{\sin\alpha - \mu\cos\alpha}\right) = \frac{5}{2}R\left(\frac{\sin\alpha + \mu\cos\alpha}{\sin\alpha - \mu\cos\alpha}\right).$$