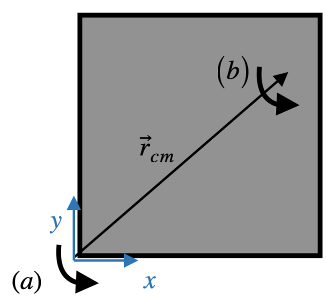
HW 10

1 Moment of Inertia

Calculate the moment of inertia of the following objects:

1. A sphere of radius R and density ρ inside of which there is a spherical cavity of radius r, relative to the three principle axes (x, y, z).
2. A rectangular flat disk with length side a and the following surface mass density $\sigma = kx^2y$.
   (a) Relative to to bottom left corner (rotation around that corner).
   (b) Next, find the center of mass of the disk and calculate its moment of inertia relative to its center of mass.



Solution:

1. Let's solve this problem for two bodies moving together:
   The whole sphere (without cavity) and the cavity (sphere with density $-\rho$ centered at distance $R - r$ from the center of the big sphere).
   First we notice that for a spherical body with uniform density (check it!):

$$I_x = I_y = I_z = \frac{2}{5}R^5\frac{4}{3}\pi\rho$$

   The body here is a symmetric top ($I_x = I_y \neq I_z$).
   The principle axis of rotation passes trough the center of mass, which is located on the axis of symmetry.
   Denote z axis to be the axis of symmetry - passes trough the center of the whole sphere and the center of the spherical cavity.
   Therefore

$$I_z = (I_z)_{sphere} + (I_z)_{cavity} \Rightarrow$$
$$I_z = \frac{4}{3}\pi\rho\frac{2}{5}\left(R^5 - r^5\right)$$

Weighted average for the COM

$$R_{COM} = \frac{M_{sphere}R_{sphere} + M_{cavity}R_{cavity}}{M_{sphere} + M_{cavity}} = \frac{\frac{4}{3}\pi R^3\rho \cdot 0 - \frac{4}{3}\pi r^3\rho \cdot (R-r)}{\frac{4}{3}\pi R^3\rho - \frac{4}{3}\pi r^3\rho} =$$

$$R_{COM} = -\frac{r^3\,(R-r)}{(R^3 - r^3)}\hat{z}$$

Now, for $I_{x/y}$ we use the parallel axis theorem to move each body's moment of inertia to the axis which passes trough the COM

$$(I_x)_{sphere} = \frac{2}{5}R^5\frac{4}{3}\pi\rho + \left(\frac{r^3\,(R-r)}{(R^3-r^3)}\right)^2\frac{4}{3}\pi\rho R^3$$

In the same manner

$$(I_x)_{cavity} = \frac{2}{5}r^5\frac{4}{3}\pi\,(-\rho) + \left(\frac{r^3\,(R-r)}{(R^3-r^3)} + (R-r)\right)^2\frac{4}{3}\pi\,(-\rho)\,r^3$$

$$(I_x)_{cavity} = -\frac{2}{5}r^5\frac{4}{3}\pi\rho - \left(\frac{R^3\,(R-r)}{(R^3-r^3)}\right)^2\frac{4}{3}\pi\rho r^3$$

Summing up the two

$$I_x = I_y = \frac{4}{3}\pi\rho\left[\frac{2}{5}\left(R^5 - r^5\right) + \frac{(R-r)^2}{(r^3-R^3)^2}\left(R^6r^3 - r^6R^3\right)\right] \Rightarrow$$

$$I_x = I_y = \frac{4}{3}\pi\rho\left[\frac{2}{5}\left(R^5 - r^5\right) - \frac{r^3R^3\,(R-r)^2}{R^3-r^3}\right]$$

2. Let us calculate the moment of inertia for the disk using $I = \int r^2 dm$, we will choose cartesian coordinate system,

(a) Relative to the bottom left corner we choose the origin at the bottom left corner so that

$$\begin{aligned} I = \int r^2 dm &= \int \left(x^2+y^2\right)\sigma\,(x,y)\,dS \\ &= \int_0^a\int_0^a \left(x^2+y^2\right)kx^2y\,dxdy \\ &= ka^7\left(\frac{1}{10}+\frac{1}{12}\right) \\ &= \frac{11}{60}ka^7. \end{aligned}$$

(b) The center of mass is

$$\begin{aligned} \boldsymbol{r}_{cm} = \frac{\int \boldsymbol{r}dm}{\int dm} &= \frac{\int_0^a\int_0^a \left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\right)kx^2y\,dxdy}{\int_0^a\int_0^a kx^2y\,dxdy} \\ &= \frac{\int_0^a\int_0^a \left(x^3y\hat{\boldsymbol{x}} + x^2y^2\hat{\boldsymbol{y}}\right)\,dxdy}{\int_0^a\int_0^a x^2y\,dxdy} \\ &= \frac{a^6\left(\frac{1}{8}\hat{\boldsymbol{x}} + \frac{1}{9}\hat{\boldsymbol{y}}\right)}{a^5/6} \\ &= 6a\left(\frac{1}{8}\hat{\boldsymbol{x}} + \frac{1}{9}\hat{\boldsymbol{y}}\right). \end{aligned}$$

Finally, we use Steiner's theorem $I = I_{cm} + d^2M$ to find the moment of inertia relative to the center of mass, using the one we found relative to the left corner:

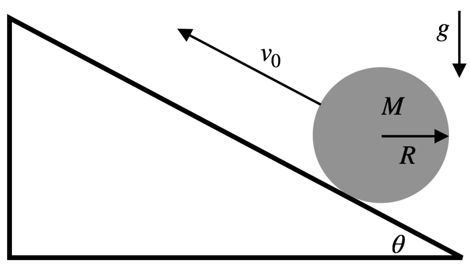
$$d^2 = |r_{cm}|^2 = a^2\left(\frac{9}{16} + \frac{36}{81}\right) \approx a^2,$$

$$M = \int dm = \frac{ka^5}{6}$$

$$I_{cm} = I_{\text{corner}} - d^2M = \frac{11}{60}ka^7 - \frac{1}{6}ka^7 = \frac{1}{60}ka^7.$$

2 Rolling cylinder

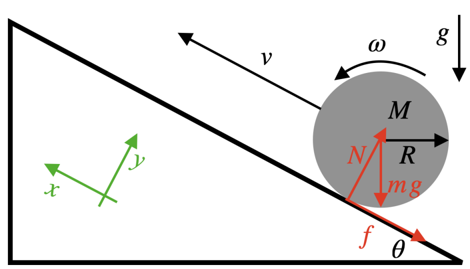
A uniformly filled cylinder with radius R and mass M is located on a plane with slope $\theta = \pi/4$. The kinetic and static friction coefficients are μ_k and μ_s, respectively. It is given that the center of mass of the cylinder moves with initial velocity v_0 and no angular velocity (relative to the center of mass). The moment of inertia of a full cylinder is $I = \frac{1}{2}MR^2$.
It is also given that μ_s is extremely large.



1. How long does it take for the cylinder to roll without slipping? What will be the velocity at that time? Assume that roll without slipping is possible, and that $\mu_k = 1$.
2. What is the change in the gravitational potential energy of the cylinder between the beginning of motion and the time when the cylinder rolls without slipping?
3. What is the total energy of the cylinder at the time when it begins rolling without slipping? Assume that the initial potential energy is $E_g = 0$.

Solution:

1. Let us first mark all the forces on the cylinder



Roll

without slipping is characterized by $v = \omega R$. Writing the force equations yields, using coordinate system parallel to the plain,

$$\begin{aligned} N - Mg\cos\theta = 0 \quad &\rightarrow \quad N = Mg\cos\theta, \\ -f - Mg\sin\theta = Ma \quad &\rightarrow \quad a = -g\left(\mu_k\cos\theta + \sin\theta\right), \end{aligned}$$

hence

$$v\left(t\right) = v_0 - gt\left(\mu_k\cos\theta + \sin\theta\right).$$

The torque equation (relative to the center of mass) reads

$$fR = I\alpha \quad \rightarrow \quad \alpha = 2\mu_k\frac{g}{R}\cos\theta,$$

hence

$$\omega\left(t\right) = 2\mu_k\frac{g}{R}t\cos\theta.$$

Plugging the expressions for ω and v into the no slipping condition yields

$$v_0 - gt'\left(\mu_k\cos\theta + \sin\theta\right) = 2\mu_k gt'\cos\theta \quad \rightarrow \quad t' = \frac{v_0}{g\left(3\mu_k\cos\theta + \sin\theta\right)}.$$

Using $\mu_k = 1$ and $\theta = \pi/4$ we find $t' = \frac{\sqrt{2}v_0}{4g}$.
The center of mass velocity at this time is

$$v\left(t'\right) = \frac{2\mu_k\cos\theta}{3\mu_k\cos\theta + \sin\theta}v_0 = \frac{v_0}{2}.$$

2. The cylinder will climb to height $h = \Delta x\left(t'\right)\sin\theta$, thus

$$\begin{aligned} h &= \left(v_0 t' - \frac{1}{2}gt'^2\left(\mu_k\cos\theta + \sin\theta\right)\right)\sin\theta \\ &= \left(\frac{v_0^2}{g\left(3\mu_k\cos\theta + \sin\theta\right)} - \frac{1}{2}\frac{v_0^2}{g\left(3\mu_k\cos\theta + \sin\theta\right)^2}\left(\mu_k\cos\theta + \sin\theta\right)\right)\sin\theta \\ &= \frac{v_0^2}{2g}\frac{5\mu_k\cos\theta + \sin\theta}{\left(3\mu_k\cos\theta + \sin\theta\right)^2}\sin\theta \\ &= \frac{3}{16}\frac{v_0^2}{g}. \end{aligned}$$

The difference in potential energy is therefore $\Delta U_g = Mgh = \frac{3}{16}Mv_0^2$.

3. The total energy at t' is

$$\begin{aligned} E_{\text{total}} &= \frac{1}{2}Mv^2\left(t'\right) + \frac{1}{2}I\omega\left(t'\right) + \Delta U_g \\ &= \frac{1}{2}M\frac{v_0^2}{4} + \frac{1}{2}\frac{M}{2}\frac{v_0^2}{4} + \frac{3}{16}Mv_0^2 \\ &= \frac{3}{8}Mv_0^2. \end{aligned}$$

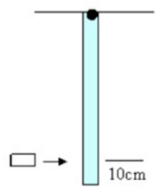
3 Bullet and Rod Collision

A wooded rod of length $L = 2\,meters$ and its mass $M = 5\,kg$ is hanging vertically from some fixed point.
A bullet of mass $m = 10\,grams$ is shot at $v = 400\,\frac{meters}{second}$ toward the rod $d = 10\,cm$ from its bottom.

1. Calculate (explicitly) the rod moment of inertia around the fixed point.

2. What would be the angular velocity of the rod right after the bullet stick to it?

3. What would be the maximal angle the rod will deviate to it?



Solution:

1. $I = \int r^2 dm$, where $r = x$, $dm = \lambda dx$, $\lambda = \frac{M}{L}$, $x : 0 \to L$.

$$I = \lambda \int_0^L x^2 dx = \frac{M}{3L} L^3 = \frac{1}{3} M L^2$$

$$I = 6.6 \left[kg \cdot m^2\right]$$

2. The angular momentum in the system conserved (no external torque)

$$mv(L-d) = \tilde{I} \omega_{col}$$

where ω_{col} is the angular momentum right after collision, and the moment of inertia for the whole system is the sum of moments and is given by

$$\tilde{I} = I + m(L-d)^2 = 6.7 \left[kg \cdot m^2\right]$$

$$6.7 \left[kg \cdot m^2\right] \omega_{col} = 7.6 \left[kg \cdot \frac{m^2}{sec}\right]$$

$$\omega_{col} = 1.13 \left[sec^{-1}\right]$$

3. The energy in the system conserved (the fixed point's force does not do work)-

$$E_i = E_f$$

$$\frac{1}{2} \tilde{I} \omega_{col}^2 = \frac{1}{2}(m+M) \tilde{I} \omega^2 + (m+M) gh$$

Finding the system C.O.M right after collision

$$\boldsymbol{R_{COM}} = \frac{M\left(-\frac{L}{2}\right) + m(d-L)}{M+m} \hat{y} \simeq -\frac{L}{2} \hat{y}$$

For the maximal angle $\theta_{max} \Rightarrow \omega = 0$ and $h = \frac{L}{2}(1 - \cos\theta_{max})$.

$$\frac{1}{2} \tilde{I} \omega_{col}^2 = (m+M) g \frac{L}{2} (1 - \cos\theta_{max}) \cong \frac{gML}{2} (1 - \cos\theta_{max})$$

$$\cos\theta_{max} = 1 - \frac{\tilde{I} \omega_{col}^2}{gML} = 0.97$$

$$\theta_{max} = 0.2 \, [rad]$$

4 Oxygen Supply

An oxygen tube is connected to the oxygen supply spacecraft: a massless rod of length D, which is perpendicular to the direction of motion of the spacecraft.
A customer called and asked that the spacecraft supply oxygen to his research team.
To do this, the oxygen spacecraft moves toward the research spacecraft at velocity v, at a horizontal distance D from the research spacecraft, so that the tip of the oxygen tube exactly connects to the research spacecraft
Once the spacecrafts pass each other, the research spacecraft crew fixes the tip of the tube to their spacecraft.
The mass of the oxygen spacecraft is M_1 and the mass of the research spacecraft is M_2.

1. What will be the angular velocity of the spacecraft after the connection?
2. Is the energy conserved during the connection?

   At some point the oxygen tube is being pulled into the oxygen spacecraft, so the distance of the tube shortens to $\frac{D}{2}$.
3. what will be the angular velocity now?
4. How much work W needed in order to pull the tube?

Solution:

Throughout the motion, there are no external forces or torques, so the linear momentum and angular momentum are conserved.
We will determine our point of reference to the center of mass of the system of the two connected spacecraft.
The distance of the center of mass from the oxygen spacecraft:

$$d_1 = \frac{M_1 \cdot 0 + M_2 \cdot D}{M_1 + M_2} = \frac{M_2}{M_1 + M_2} D$$

So the distance of the center of mass from the research spacecraft:

$$d_2 = D - d_1 = \frac{M_1}{M_1 + M_2} D$$

1. Using Angular momentum conservation

$$L_i = m = M_1 v \frac{M_2}{M_1 + M_2} D$$

After the connection the system move rigidly

$$L_f = I\omega$$

where the moment of inertia around the center of mass

$$I = M_1 \left(\frac{M_2}{M_1 + M_2} D \right)^2 + M_2 \left(\frac{M_2}{M_1 + M_2} D \right)^2$$

$$I = \frac{M_1 M_2}{M_1 + M_2} D^2$$

$$L_i = L_f$$

$$v \frac{M_1 M_2}{M_1 + M_2} D = \frac{M_1 M_2}{M_1 + M_2} D^2 \omega$$

$$\omega = \frac{v}{D}$$

*Could have been guessed from units consideration!
2.Before the connection, there was only a kinetic energy

$$E_i = \frac{1}{2} M_1 v^2$$

After the connection, the c.o.m. has a velocity that can be found from linear momentum conservation

$$(M_1 + M_2)\, v_{com} = M_1 v$$

$$v_{com} = \frac{M_1}{M_1 + M_2} v$$

So

$$\begin{aligned} E_f &= \frac{1}{2}(M_1 + M_2)\, v_{com}^2 + \frac{1}{2} I \omega^2 = \\ &= \frac{1}{2}(M_1 + M_2) \frac{M_1^2}{(M_1 + M_2)^2} v^2 + \frac{1}{2} \frac{M_1 M_2}{M_1 + M_2} D^2 \left(\frac{v}{D}\right)^2 = \\ &= \frac{1}{2} \frac{M_1^2 + M_1 M_2}{M_1 + M_2} v^2 = \frac{1}{2} \frac{M_1 + M_2}{M_1 + M_2} M_1 v^2 = \frac{1}{2} M_1 v^2 = E_i \end{aligned}$$

We got that the energy conserved.
3. The forces that pull the tube are internal forces - Linear and Angular momentum still conserved.

$$I\omega = I_2 \omega_2$$

The new distances to the c.o.m are

$$d_1 = \frac{M_2}{M_1 + M_2} \frac{D}{2} \quad d_2 = \frac{M_1}{M_1 + M_2} \frac{D}{2}$$

And the new moment of inertia

$$I_2 = M_1 \left(\frac{M_2}{M_1 + M_2} \frac{D}{2} \right)^2 + M_2 \left(\frac{M_2}{M_1 + M_2} \frac{D}{2} \right)^2 = \frac{I}{4}$$

then

$$\omega_2 = 4\omega = 4\frac{v}{D}$$

4. Let's see how much the energy was changed (v_{com} does not changed)

$$W = \Delta E = \frac{1}{2} I_2 \omega_2^2 - \frac{1}{2} I \omega^2 = \frac{1}{2} I \omega^2 \left(\frac{1}{4} 4^2 - 1 \right) = \frac{3}{2} I \omega^2$$

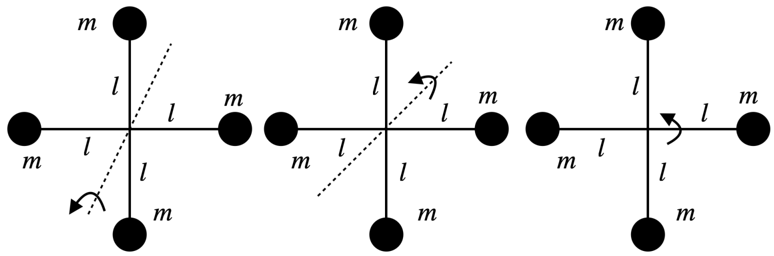
$$W = \frac{3}{2} \frac{M_1 M_2}{(M_1 + M_2)} v^2$$

Imagine yourself in a spaceship attached to another spacecraft as you move around as one body.
What's harder? Pull the other spacecraft towards you or release the tube and move the other spacecraft away?
Hint: in 4 we got $W > 0$.

5 Four Point-like bodies

Four point-like bodies with mass m each are arranged on a solid massless cross. The distances between the bodies and the center is l.

1. What is the moment of inertia relative to the following axes:



   (a) One that goes through the center of the cross, perpendicular to it.
   (b) One that is on the plane of the cross and forms 45° with the cross.
   (c) One that is on the plane of the cross and goes through bodies 2 and 4.

2. Show that for rotation with angular velocity ω in configuration (a), the sum of the linear kinetic energy of the bodies equals to that of the angular kinetic energy of the cross $\left(\sum \frac{1}{2}mv^2 = \frac{1}{2}I\omega^2\right)$.

3. What would be the angular momentum if *suddenly* the cross disappears and the bodies move freely with the velocities they had just before it was gone.

Solution:

1. Following $I = \sum_i m_i r_i^2$ we find

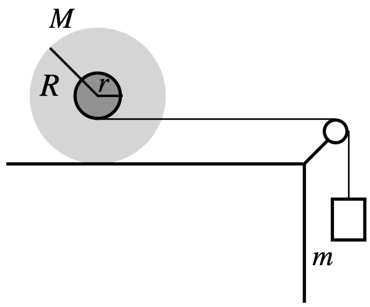
   (a) The distance of all bodies from the axis is l, thus
   $$I_a = 4ml^2.$$
   (b) The distance of all bodies from the axis is $l \sin 45° = l/\sqrt{2}$, thus
   $$I_b = 2ml^2.$$
   (c) Only 2 of the bodies affect the moment of inertia (the other two are on the axis of rotation) and are at distance l from it, thus
   $$I_c = 2ml^2.$$

2. The velocity of each body is $v = \omega l$, hence the linear kinetic energy of the bodies is
$$(E_k)_{\text{linear}} = 4 \times \frac{1}{2}mv^2 = 2m\omega^2 l^2,$$
while the angular kinetic energy is
$$(E_k)_{\text{angular}} = \frac{1}{2}I_a\omega^2 = 2m\omega^2 l^2,$$
therefore $(E_k)_{\text{linear}} = (E_k)_{\text{angular}}$.

3. For sudden disappear, involving no external forces, angular and linear momentum are conserved. Therefore
$$L = I_a\omega = 4ml^2\omega,$$
and the velocities of each body will be in the direction of $\hat{\boldsymbol{\theta}}$ and of value $v = \omega l$.

6 Rolling Weight - Bonus

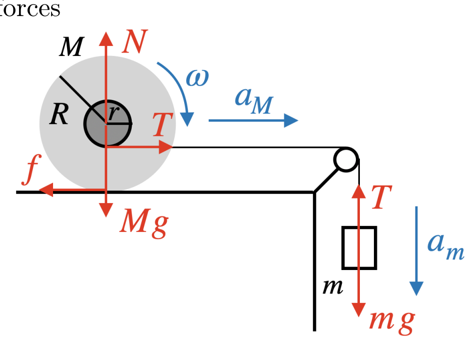
A point mass m is connected to to a uniform cylinder with radius R, mass M and a given moment of inertia $I = kMR^2$, through a massless ideal rope, as shown in the figure. The rope is connected to the cylinder at a socket with radius r (see figure). Assume that the friction coefficient between the cylinder and the floor is extremely large.



1. What is the acceleration of the mass m? What is the linear acceleration of mass M?
2. What is the velocity of mass m after it fell distance h? **Use energy considerations.** Assume that the entire system starts at rest.

Solution:

1. Let us begin by marking all the forces



Next, let us write down the force equation for the point mass

$$T - mg = -ma_m,$$

while for the cylinder we find

$$T - f = Ma_M,$$
$$N - Mg = 0.$$

The cylinder also has a relevant torque equation (relative to the center of mass)

$$fR - Tr = I\alpha,$$

where the angular acceleration follows $\alpha R = a_M$, since there is no slipping. We also note that

$$y_m = x_M - r\theta \quad \rightarrow \quad a_m = a_M - r\alpha \quad \rightarrow \quad a_m = a_M\left(1 - \frac{r}{R}\right),$$

due to the tension and constant length of the rope.

Thus, we have 4 unknowns: N, T, f and a with 4 equations. Solving for a_M, we start with eliminating $T = f + Ma_M$ in the cylinders force and torque equations, using the point mass force equation,

$$f = mg - \left[m\left(1 - \frac{r}{R}\right) + M\right] a_M,$$
$$f = \frac{kR + r}{R - r} Ma_M,$$

where we expressed a_m in terms of a_M (as described above). Subtracting these two equations and solving for a_M gives us

$$a_M = \frac{mgR\left(R - r\right)}{m\left(R - r\right)^2 + \left(1 + k\right)R^2 M}.$$

Using the acceleration relation we found yields

$$a_m = \frac{mg\left(R - r\right)^2}{m\left(R - r\right)^2 + \left(1 + k\right)R^2 M}.$$

Short dimension check shows that the $(\text{length})^2$ and mass cancel for both expressions on the right hand side and the dimensions are of acceleration. We also note that for $r = R$ the accelerations vanish, which is plausible since the tension and friction are exerted at the same point and we assumed no slipping, i.e. friction could always match the exerted tension.

2. Since the friction force does no work, we have energy conservation. Let us choose the gravitational potential energy of mass m at the final position (i.e. after it fell down distance h), so that the initial energy of the system is the potential energy alone

$$E_i = mgh.$$

While the final energy of the system is all kinetic

$$E_f = \frac{1}{2}Mv_M^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}mv_m^2,$$

where the velocities are related by the same rules as the accelerations

$$v_M = \omega R, \quad v_m = v_M\left(1 - \frac{r}{R}\right),$$

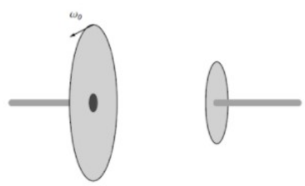
thus

$$mgh = \frac{1}{2}Mv_M^2 + \frac{1}{2}kMv_M^2 + \frac{1}{2}mv_m^2 \quad \rightarrow \quad mgh = \frac{1}{2}MR^2 v_m^2 \frac{(1+k)}{\left(R - r\right)^2} + \frac{1}{2}mv_m^2,$$

$$v_m = \sqrt{\frac{2mgh}{M\left(\frac{R}{R-r}\right)^2 (1+k) + m}}.$$

7 Clutch - Bonus

Two discs with moment of inertia of $I_1$and I_2.
At $t = 0$ the first disc is turning with frequency ω_0.
Bring the two discs closer together until they touch each other.
As a result of the friction force the two discs eventually reach a common frequency Ω.
Find Ω. How much energy lost in the process?



Solution:

The friction is internal - angular momentum is conserved.

$$I_1\omega_0 = (I_1 + I_2)\,\Omega$$
$$\Omega = \frac{I_1}{I_1 + I_2}\omega_0$$

The energy difference is

$$\begin{aligned} E_i - E_f &= \frac{1}{2}I_1\omega_0^2 - \frac{1}{2}(I_1 + I_2)\,\Omega^2 = \\ &= \frac{1}{2}I_1\omega_0^2 - \frac{1}{2}(I_1 + I_2)\left(\frac{I_1}{I_1 + I_2}\omega_0\right)^2 = \\ &= \frac{1}{2}I_1\omega_0^2\left(1 - \frac{I_1}{I_1 + I_2}\right) = \underline{\frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\omega_0^2}. \end{aligned}$$

8 Bicycle - Bonus

Assume that the center of mass of a bicycle and rider is at height $2l$ above the ground.
Each wheel has mass m, radius l and moment of inertia ml^2.
The bicycle moves with velocity V in a circular path of radius $R \gg l$.
Show that it leans through an angle given by $\tan\alpha = \frac{V^2}{Rg}\left(1 + \frac{m}{M}\right)$, where M is the **total** mass.

Solution:

Let's choose polar coordinates system in the frame rotating with angular velocity $\Omega = \frac{V}{R}$ around the center of the bicycle circular path.
The forces acting on our system (rider and bike) is the gravitational force and the centrifugal force, both acting on the center of mass.

$$\boldsymbol{F_g} = -Mg\hat{\boldsymbol{z}}$$
$$\boldsymbol{F_c} = M\Omega^2 R\hat{\boldsymbol{r}}$$

The magnitude of the angular momentum of the two spinning wheels:

$$L = 2I\omega = 2ml^2\omega$$

If the biker is leaning with an angle α (the angle between the vertical direction and the bike) then the angular momentum vector of the bike is

$$\boldsymbol{L} = 2ml^2\omega\left(\cos\alpha\hat{\boldsymbol{r}} + \sin\alpha\hat{\boldsymbol{z}}\right).$$

Note that there is a connection between the frequencies

$$\Omega R = \omega l$$

In our system $\frac{d}{dt}\hat{z} = 0$ and $\frac{d}{dt}\hat{r} = \Omega\hat{\varphi}$. then:

$$\frac{d}{dt}\boldsymbol{L} = \Omega L_r \hat{\varphi} = 2mRl\Omega^2 \cos\alpha\hat{\varphi}$$

For the torques acting on the bike we first need to express the vector pointing from the floor to the c.o.m.

$$\boldsymbol{d} = 2l\left(-\sin\alpha\hat{\boldsymbol{r}} + \cos\alpha\hat{\boldsymbol{z}}\right)$$

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{d} \times (\boldsymbol{F_g} + \boldsymbol{F_c}) = 2l\left(-\sin\alpha\hat{\boldsymbol{r}} + \cos\alpha\hat{\boldsymbol{z}}\right) \times M\left(\Omega^2 R\hat{\boldsymbol{r}} - g\hat{\boldsymbol{z}}\right) = \\ &= 2lM\left(g\sin\alpha - \Omega^2 R\cos\alpha\right)\hat{\boldsymbol{\varphi}}\end{aligned}$$

Using

$$\frac{d}{dt}\boldsymbol{L} = \boldsymbol{\tau}$$

$$2mRl\Omega^2\cos\alpha = 2lM\left(g\sin\alpha - \Omega^2 R\cos\alpha\right)$$

$$1 = \frac{M}{m}\left(\frac{g}{R\Omega^2}\tan\alpha - 1\right)$$

Setting in $\Omega = \frac{V}{R}$ and some more algebra

$$\tan\alpha = \frac{V^2}{Rg}\left(1 + \frac{m}{M}\right)$$