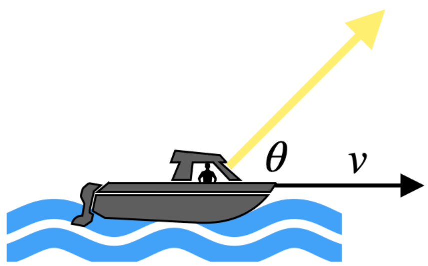
HW 12

1 Relativistic angle

A man on a boat, which is moving at speed v, lights the sky with a flashlight, so that the angle between the light beam and the boat is θ' (as observed in the boat's frame). An observer on the shore is measuring the angle between the light beam and the boat to be θ.



1. What is the relation between θ and θ'?
2. Show that, in the observer's frame, the speed of light equals c as well.

Solution:

1. Since the light is actually photons moving relative to the boat, we must use velocity transformations in order to find the direction of their velocity relative to the observer. Denoting the velocity of the light beam by u' (relative to the boat) and the direction of motion by x, we may write

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x},$$
$$u_y = \frac{u'_y}{\gamma \left(1 + \frac{v}{c^2} u'_x\right)}.$$

   Therefore, the angle in the observer's frame is

$$\tan\theta = \frac{u_y}{u_x} = \frac{u'_y}{\gamma\left(u'_x + v\right)} = \frac{1}{\gamma\left(1 + \frac{v}{u'_x}\right)} \tan\theta' = \frac{1}{\gamma\left(1 + \frac{v}{u'\cos\theta'}\right)} \tan\theta'.$$

2. The velocity in the frame of the observer is, plugging in $u' = c$,

$$u^2 = u_x^2 + u_y^2 = \frac{\left(u'_x + v\right)^2 + u'^2_y\left(1 - v^2/c^2\right)}{\left(1 + \frac{v}{c^2}u'_x\right)^2}$$
$$= \frac{\left(c\cos\theta' + v\right)^2 + c^2\sin^2\theta'\left(1 - v^2/c^2\right)}{\left(1 + \frac{v}{c}\cos\theta'\right)^2}$$
$$= \frac{c^2\cos^2\theta' + 2cv\cos\theta' + v^2 + c^2\sin^2\theta' - v^2\sin^2\theta'}{1 + 2\frac{v}{c}\cos\theta' + \frac{v^2}{c^2}\cos^2\theta'}$$
$$= c^2\frac{1 + 2\frac{v}{c}\cos\theta' + \frac{v^2}{c^2}\left(1 - \sin^2\theta'\right)}{1 + 2\frac{v}{c}\cos\theta' + \frac{v^2}{c^2}\cos^2\theta'}$$
$$= c^2,$$

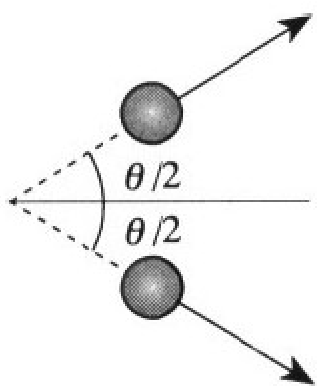
which is expected, after all we used the assumption that c is the same in all frames to deduce the velocity transformations.

2 Proton-Proton Collision

A proton with $\gamma = 1/\sqrt{1 - (v^2/c^2)}$ collide elastically with a proton at rest.
If the two protons rebound with equal energies, what will be the angle θ between them?

Solution:

Since the energies of the protons after the collision are equal, they will rebound at the same angle $\frac{\theta}{2}$ relative to the initial momentum of the proton (see figure).



Before the collision

$$p = \gamma m v = \gamma m c\frac{v}{c}$$

$$E = mc^2 + \sqrt{\left(pc\right)^2 + \left(mc^2\right)^2} = mc^2\left(1 + \sqrt{\gamma^2\left(\frac{v}{c}\right)^2 + 1}\right) =$$
$$= mc^2\left(1 + \sqrt{\frac{\left(\frac{v}{c}\right)^2}{1 - (v^2/c^2)} + 1}\right) = mc^2\left(1 + \sqrt{\frac{1}{1 - (v^2/c^2)}}\right) = \left(1 + \gamma\right)mc^2$$

Using momentum conservation, we have

$$\gamma m v = 2\tilde{\gamma} m \tilde{v}\cos\frac{\theta}{2}$$

where $\tilde{\gamma}$ and $\tilde{v}$ are the Lorentz factor and the velocity after the collision.
Energy conservation yields

$$(1+\gamma)\, mc^2 = 2\tilde{\gamma} mc^2$$

or

$$1+\gamma = 2\tilde{\gamma}$$

Using the definition of Lorentz factor

$$\tilde{\gamma} = 1/\sqrt{1-(\tilde{v}^2/c^2)}$$

$$\tilde{\gamma}^2 = \frac{1}{1-(\tilde{v}^2/c^2)}$$

$$c^2 - \tilde{v}^2 = \frac{c^2}{\tilde{\gamma}^2}$$

$$\tilde{v} = c\frac{\sqrt{\tilde{\gamma}^2-1}}{\tilde{\gamma}} = c\frac{\sqrt{(1+\gamma)^2-4}}{1+\gamma}$$

In the same manner

$$v = c\frac{\sqrt{\gamma^2-1}}{\gamma}$$

We obtain from the momentum equation

$$\gamma v = 2\tilde{\gamma}\tilde{v}\cos\frac{\theta}{2}$$

$$\cancel{\gamma}\cancel{c}\frac{\sqrt{\gamma^2-1}}{\cancel{\gamma}} = \cancel{(1+\gamma)}\cancel{c}\frac{\sqrt{(1+\gamma)^2-4}}{\cancel{1+\gamma}}\cos\frac{\theta}{2}$$

$$\sqrt{\gamma^2-1} = \sqrt{(1+\gamma)^2-4}\cos\frac{\theta}{2}$$

$$\gamma^2 - 1 = \left[(1+\gamma)^2 - 4\right]\cos^2\frac{\theta}{2}$$

$$\gamma^2 - 1 = \left[\gamma^2 + 2\gamma - 3\right]\cos^2\frac{\theta}{2}$$

$$(\gamma+1)\cancel{(\gamma-1)} = (\gamma+3)\cancel{(\gamma-1)}\cos^2\frac{\theta}{2}$$

Using the trigonometric identity

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\frac{\gamma+1}{\gamma+3} - 1 = \frac{2\gamma+2-\gamma-3}{\gamma+3}$$

$$\cos\theta = \frac{\gamma-1}{\gamma+3}$$

For $\gamma \approx 1$, i.e. in the classical limit pf low velocity, $\cos\theta = 0$, and we obtain the familiar result that the angle between billiard balls rebounding with equal energy is $\frac{\pi}{2}$.
If $\gamma \gg 1$ (extremely relativistic case), then $\cos\theta \approx 1$ and $\theta \to 0$.

3 Relativistic Force

A particle with mass of m_0 moves in 1D space.
Its motion is given by $x(t) = \sqrt{b^2 + c^2t^2} - b$, where $b = const$ and c is the speed of light.
Find the force acting on the particle.

Solution:

One can obtain the force acting on a particle using Newton's second law

$$\frac{dp}{dt} = F$$

Where for a relativistic particle the momentum is given by (using scalars instead of vectors in 1D)

$$p = \gamma m_0 v$$

and the derivation by time of the momentum is given by

$$\dot{p} = \dot{\gamma} m_0 v + \gamma m_0 \dot{v}$$

calculating the derivation by time of Lorentz factor

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -\frac{\cancel{1}}{\cancel{2}}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\cancel{-2}\frac{v}{c^2}\dot{v}\right) = \gamma^3\frac{v\dot{v}}{c^2}$$

so

$$\dot{p} = \gamma m_0 \dot{v}\left(\gamma^2\frac{v^2}{c^2}+1\right) = \gamma m_0 \dot{v}\left(\frac{v^2/c^2}{1-v^2/c^2}+1\right) = \gamma m_0 \dot{v}\left(\frac{1}{1-v^2/c^2}\right) = \gamma^3 m_0 \dot{v}$$

Finding $\dot{v}$

$$v = \dot{x} = \frac{c^2 t}{\sqrt{b^2 + c^2t^2}}$$

$$\dot{v} = \frac{c^2\left(b^2 + c^2t^2\right) - c^4t^2}{(b^2 + c^2t^2)^{\frac{3}{2}}} = \frac{c^2b^2}{(b^2 + c^2t^2)^{\frac{3}{2}}}$$

and

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1 - \frac{c^2t^2}{b^2 + c^2t^2}\right)^{-\frac{1}{2}} = \left(\frac{b^2}{b^2 + c^2t^2}\right)^{-\frac{1}{2}}$$

Finally we get

$$F = \gamma^3 m_0 \dot{v} = \left(\frac{b^2}{\cancel{b^2 + c^2t^2}}\right)^{-\frac{3}{2}} m_0 \left(\frac{c^2b^2}{\cancel{(b^2 + c^2t^2)}}\right) = m_0\frac{c^2b^2}{b^3} = \frac{1}{b}m_0c^2$$

4 Relativistic Newton's Second Law

Consider the relativistic form of Newton's second law.
Show that when $\overrightarrow{F}$ is parallel to $\overrightarrow{v}$

$$F = m\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\frac{dv}{dt}$$

Solution:

Newton's second law state that

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Where for a relativistic particle the momentum is given by

$$\vec{p} = \gamma m \vec{v}$$

In order to know $\frac{d\vec{p}}{dt}$ we first need to calculate $\frac{d\vec{v}}{dt}$ and $\frac{d\gamma}{dt}$.

$$\frac{d\vec{v}}{dt} = \underbrace{\frac{dv}{dt}\hat{v}}_{tangential} + \underbrace{v\frac{d\hat{v}}{dt}}_{normal}$$

One also can find that

$$\frac{d\gamma}{dt} = \gamma^3 m \frac{v^2}{c^2}\frac{dv}{dt}$$

If the force is parallel to the velocity we can write $\vec{F} = F\hat{v}$, and Newton's second law becomes

$$F\hat{v} = \gamma^3 m \frac{v^2}{c^2}\frac{dv}{dt}\hat{v} + \gamma m \frac{dv}{dt}\hat{v} + \gamma m v \frac{d\hat{v}}{dt}$$

and we conclude

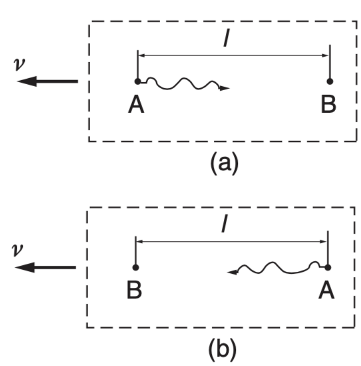
$$\frac{d\hat{v}}{dt} = 0$$

and

$$F = \gamma m \frac{dv}{dt}\left(\gamma^2\beta^2 + 1\right) = \gamma^3 m \frac{dv}{dt} = m\left(1 - v^2/c^2\right)^{-\frac{3}{2}}\frac{dv}{dt}.$$

5 One-way test of the constancy of c

Light in a Michelson–Morley interferometer makes a roundtrip, and the predicted time delay is second order, proportional to v^2/c^2. Here is an experiment that would give a first-order result proportional to v/c. Consider a laboratory moving through the ether with speed v in the direction shown. The observers have clocks and light pulsers. At time $t = 0$ A sends a signal to B a distance l away, sketch (a). B records the arrival time. The laboratory is then rotated 180°, reversing the positions of A and B. At time $t = T$, A sends a second signal to B, sketch (b).
Note: the ether theorem states that light moves with velocity c relative to the ether.



1. Show that according to the ether theory, the interval that B observes between the signals is $T + \Delta T$, where

$$\Delta T \approx \frac{2l}{c}\frac{v}{c}$$

correct to order $(v/c)^3$.

2. Assume that one clock in this experiment is on the ground and the other is in a satellite overhead. For a circular orbit with a period of 24 hours, $l = 5.6R_e$, where $R_e = 6.4 \times 10^6$m is the Earth's radius. Using an atomic clock stable to within 1 part in 10^{16}, what is the smallest value of v this experiment could detect?

Solution:

1. At time $t = 0$ A sends a signal to B, which arrives to B at

$$t_a = \frac{l}{c - v} \approx \frac{l}{c}\left(1 + \frac{v}{c}\right).$$

At time T a signal is sent from B, reaching A at

$$t_b = T + \frac{l}{c + v} \approx \frac{l}{c}\left(1 - \frac{v}{c}\right).$$

We find that the time difference we might expect between the signals is

$$\Delta T = |t_b - t_a - T| = \frac{2l}{c}\frac{v}{c}.$$

2. In this apparatus, the distance is $l = 4.6R_e$ and the time it takes to rotate it a 180° is $T = 12$ hours, thus $\Delta T =$ (time of the experiment) $\times$ (clock stability fraction) $= \left(12\text{hours} \times 3600\frac{\text{sec}}{\text{hour}}\right) \times 10^{-16} = 4.3 \times 10^{-12}$s. The minimum velocity that such apparatus could detect is

$$v = \frac{c^2 \Delta T}{2 \times 4.6 R_e} = \frac{\left(3 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2 \times \left(4.3 \times 10^{-12}\text{s}\right)}{2 \times 4.6 \times 6.4 \times 10^6\text{m}} = 6.6 \times 10^{-3}\frac{\text{m}}{\text{s}}.$$

6 Pole-vaulter paradox - Bonus

The pole-vaulter has a pole of length l_0, and the farmer has a barn $\frac{3}{4}l_0$ long. The farmer bets that he can shut the front and rear doors of the barn with the pole completely inside. The bet being made, the farmer asks the pole-vaulter to run into the barn with a speed $v = \sqrt{3}c/2$. In this case the farmer observes the pole to be Lorentz contracted to $l = l_0/2$, and the pole fits into the barn with ease. The farmer slams the door the instant the pole is inside, and claims the bet. The pole-vaulter disagrees: he sees the barn contracted by a factor of 2, so the pole can't possibly fit inside. Let the farmer and barn be in system S and the pole-vaulter in system S'. Call the leading end of the pole A, and the trailing end B.

1. The farmer in S sees A reach the rear door at $t_A = 0$, and closes the front door at the same time $t_A = t_B = 0$. What is the length of the pole as seen in S?

2. The pole-vaulter in S' sees A reach the rear door at t'_A . Where does the pole-vaulter see B at this instant?

3. Show that in S', A and B do not lie inside the barn at the same instant.

Solution:

1. Let us denote the coordinates of the pole in the S frame as x_A and x_B, so that the coordinates the pole-vaulter observes are

$$x'_A = \gamma\left(x_A - v\cancelto{0}{t_A}\right),$$

$$x'_B = \gamma\left(x_B - v\cancelto{0}{t_B}\right),$$

thus

$$x'_A - x'_B = l_0 = \gamma \underbrace{(x_A - x_B)}_{l} \quad \rightarrow \quad l = l_0\sqrt{1-(v/c)^2} = l_0/2,$$

the pole fits inside the barn.

2. Since the pole is stationary relative to the S' frame, the runner measures its actual length l_0, therefore if point A reaches the rear door at t'_A, then point B lies outside the barn.

3. In the S' frame

$$\begin{aligned} x'_A &= \gamma (x_A - vt_B), \\ x'_B &= \gamma (x_B - vt_B), \\ t'_A &= \gamma \left(t_A - \frac{v}{c^2}x_A\right), \\ t'_B &= \gamma \left(t_B - \frac{v}{c^2}x_B\right). \end{aligned}$$

Taking $t_A = t_B$, i.e. measuring the position of A and B simultaneously, we find

$$\begin{aligned} x'_A - x'_B &= \gamma (x_A - x_B) = \gamma l = l_0, \\ t'_A - t'_B &= \gamma \frac{v}{c^2}(x_B - x_A) = -\gamma \frac{vl}{c^2} = -\frac{v}{c^2} l_0, \end{aligned}$$

so that points A and B are not measured simultaneously - the two ends of the pole are not inside the barn at the same instant. In particular, $t'_A < t'_B$, the pole reaches the rear door before the farmer shuts the door.
In the S' frame, once the front door is closed, $x'_B = 0$ and $x'_A = l_0$ ($t'_A = t'_B$), the pole is outside the rear door. Denoting this event as C, the farmer observes C as

$$\begin{aligned} x_C &= x_A - x_B = \gamma (x'_A + vt'_A) - \gamma (x'_B + vt'_B) = \gamma l_0 = 2l_0, \\ t_C &= t_A - t_B = \gamma \left(t'_A + \frac{v}{c^2}x'_A\right) - \gamma \left(t'_B + \frac{v}{c^2}x'_B\right) = \gamma \frac{v}{c^2} l_0 = \frac{\sqrt{3}}{c} l_0. \end{aligned}$$

That is, $t_A > t_B$, event C occurs after the door is shut. Therefore, both are correct!

7 The consequences of endless acceleration - Bonus

The relativistic transformation for acceleration is easily derived from the transformation of velocities

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}},$$

$$du = \frac{du'\left(1 + \frac{u'v}{c^2}\right) - (u' + v)\, v/c^2 du'}{\left(1 + \frac{u'v}{c^2}\right)^2} = \frac{1 - v^2/c^2}{\left(1 + \frac{u'v}{c^2}\right)^2} du' = \frac{1}{\gamma^2} \frac{du'}{\left(1 + \frac{u'v}{c^2}\right)^2}.$$

Whereas

$$dt = \gamma \left(dt' + \frac{v}{c^2} dx'\right) = \gamma dt' \left(1 + \frac{vu'}{c^2}\right),$$

thus

$$a = \frac{du}{dt} = \frac{du'}{dt'} \frac{1}{\gamma^3} \frac{1}{\left(1 + \frac{u'v}{c^2}\right)^3} = a' \frac{1}{\gamma^3} \frac{1}{\left(1 + \frac{u'v}{c^2}\right)^3},$$

indicating the impossibility of accelerating a system to a velocity greater than c. Consider a spaceship that accelerates at constant rate a_0 as measured by an accelerometer carried aboard, for instance a mass stretching a spring.

1. Find the speed after time t for an observer in the system in which the spaceship was originally at rest.
2. The speed predicted classically is $v_0 = a_0 t$. What is the actual speed for the following cases: $v_0 = 10^{-3}c,\ c,\ 10^3 c$.

Solution:

1. The accelerometer is at rest in the frame for the spaceship, thus $u' = 0$ and we find

$$a = \frac{dv}{dt} = \frac{a_0}{\gamma^3} = a_0 \left(1 - \frac{v^2}{c^2}\right)^{3/2}.$$

Integrating this differential equation yields

$$\int_0^v \left(1 - \frac{v^2}{c^2}\right)^{3/2} dv = \int_0^t a_0 dt,$$

changing variables to $s = v/c$, $ds = dv/c$ yields

$$c \int_0^s \left(1 - s^2\right)^{3/2} ds = \int_0^t a_0 dt$$

$$\frac{cs}{\sqrt{1-s^2}} = a_0 t \quad \rightarrow \quad \frac{v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} a_0 t,$$

note that for $t \to \infty$, $v \to c$.

2. Using velocity transformation

$$v = \frac{u_0}{\sqrt{1 + \left(\frac{u_0}{c}\right)^2}},$$

we find that

$$\begin{aligned} u_0 = 10^{-3}c: &\quad v = \frac{10^{-3}c}{\sqrt{1+10^{-6}}} \approx 10^{-3}c, \\ u_0 = c: &\quad v = \frac{c}{\sqrt{1+1}} \approx 0.71c, \\ u_0 = 10^3 c: &\quad v = \frac{10^3 c}{\sqrt{1+10^6}} = \frac{c}{\sqrt{1+10^{-6}}} \approx \left(1 - 10^{-6}/2\right) c. \end{aligned}$$

In reality, the speed of any particle cannot exceed the speed of light.