Selected topics in solid state physics 2
Part 1

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I. TUNNEL JUNCTION

Consider a tunnel junction. The Hamiltonian reads $H = H_L + H_R + H_T$, where

$$H_T = \sum_{k,p} t_{k,p} c_k^\dagger d_p + h.c. \ . \tag{1}$$

We use the Golden Rule to calculate the rate of tunnelling from left to right and vice versa:

$$\begin{aligned}
\Gamma_{L\to R} &= \frac{2\pi}{\hbar} \sum_{k,p} |t|^2 f_L(\epsilon_k)(1 - f_R(\epsilon_p))\delta(\epsilon_k - \epsilon_p) \\
&= \frac{2\pi}{\hbar} |t|^2 \int d\epsilon\, \rho_L(\epsilon)\rho_R(\epsilon) f_L(\epsilon)(1 - f_R(\epsilon)) \\
&= \frac{2\pi}{\hbar} |t|^2 \int d\epsilon\, \rho_L(\epsilon)\rho_R(\epsilon) f(\epsilon - \mu_L)(1 - f(\epsilon - \mu_R)) \\
&= \frac{2\pi}{\hbar} |t|^2 \rho_L \rho_R \frac{\Delta\mu}{1 - e^{-\beta\Delta\mu}} \ ,
\end{aligned} \tag{2}$$

where $\Delta\mu \equiv \mu_L - \mu_R$. For simplicity we assume the absolute value of the tunnelling amplitudes to be constant, i.e., $|t_{k,p}| \equiv |t|$. We observe that the tunnelling rate is not vanishing if $\Delta\mu = 0$ and even if $\Delta\mu < 0$, if the temperature is finite $T > 0$. This is the reason for the thermal noise (see below). Identifying the voltage as $\Delta\mu = eV$ we obtain the current

$$I = e(\Gamma_{L\to R} - \Gamma_{R\to L}) = V \frac{e^2}{h} (2\pi)^2 |t|^2 \rho_L \rho_R \ . \tag{3}$$

Thus we obtain the Ohm's law $I = G_T V$, where the tunnelling conductance $G_T = 1/R_T$ is given by $G_T = g_T G_K$, where

$$g_T \equiv (2\pi)^2 |t|^2 \rho_L \rho_R \tag{4}$$

is the dimensionless tunnelling conductance and $G_K = 1/R_K = e^2/h = e^2/(2\pi\hbar)$ is the conductance quantum.

For electrons with spin we effectively have 2 parallel channels, since

$$H_T = \sum_{k,p,\sigma} t_{k,p} c_{k,\sigma}^\dagger d_{p,\sigma} + h.c. \ . \tag{5}$$

Thus we obtain $g_T = 2 \times (2\pi)^2 |t|^2 \rho_L \rho_R$, where $\rho_{L/R}$ are the orbital densities of states.

A. Noise

We write down the master equation for the probability $p(n, \tau)$ that n electrons have tunnelled during time τ:

$$\dot{p}(n) = \Gamma_{L\to R}\, p(n-1) + \Gamma_{R\to L}\, p(n+1) - \left[\Gamma_{L\to R} + \Gamma_{R\to L}\right] p(n) \ . \tag{6}$$

We employ here the simplest version of the more general method called ”full counting statistics”. In this method one calculates the generating function, which allows to find all moments and cumulants of $p(n)$. For the generating function $\chi(k) = \sum_n p(n) e^{-ikn}$ this gives

$$\dot{\chi}(k) = \Big[\Gamma_{L\to R}(e^{-ik} - 1) + \Gamma_{R\to L}(e^{ik} - 1)\Big]\, \chi(k)\ . \tag{7}$$

Taking into account the initial condition $p(n,0) = \delta_{n,0}$, i.e., $\chi(k,0) = 1$, we obtain

$$\ln \chi(k,\tau) = \tau\ \Big[\Gamma_{L\to R}(e^{-ik} - 1) + \Gamma_{R\to L}(e^{ik} - 1)\Big]\ . \tag{8}$$

The first cumulant gives then the current

$$C_1 = i\frac{\partial}{\partial k} \ln \chi(k)|_{k=0} = \tau(\Gamma_{L\to R} - \Gamma_{R\to L}) = \tau I/e\ . \tag{9}$$

The second cumulant describes noise

$$C_2 = \left(i\frac{\partial}{\partial k}\right)^2 \ln \chi(k)|_{k=0} = \tau(\Gamma_{L\to R} + \Gamma_{R\to L}) = \frac{\tau I}{e} \coth\frac{\beta e V}{2}\ . \tag{10}$$

The second cumulant is related to the zero-frequency noise of current. Indeed $e^2 C_2 = e^2(\langle n^2\rangle - \langle n\rangle^2)$. Treating the current as a classical fluctuating quantity we obtain $en(\tau) = \int_0^\tau I(\tau')d\tau'$. Thus

$$e^2 C_2 = \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \left(\langle I(\tau_1) I(\tau_2)\rangle - \langle I\rangle^2\right)\ . \tag{11}$$

We define $S_I(\tau_1,\tau_2) \equiv \langle I(\tau_1)I(\tau_2)\rangle - \langle I\rangle^2$ (in the quantum regime it will be the symmetrized correlator $S_I(\tau_1,\tau_2) \equiv (1/2)\langle I(\tau_1)I(\tau_2) + I(\tau_2)I(\tau_1)\rangle - \langle I\rangle^2$). Assuming stationarity $S_I(\tau_1,\tau_2) = S_I(\tau_1 - \tau_2)$ we obtain

$$\begin{aligned} e^2 C_2 &= \int_0^\tau dt_1 \int_0^\tau dt_2\, S_I(\tau_1 - \tau_2) = \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \int \frac{d\omega}{2\pi}\, S_I(\omega) e^{-i\omega(\tau_1-\tau_2)} \\ &= \int \frac{d\omega}{2\pi}\, S(\omega)\, \frac{\sin^2(\omega\tau/2)}{(\omega/2)^2} \approx S_I(\omega\approx 0)\,\tau\ . \end{aligned} \tag{12}$$

Thus we obtain

$$S_I \equiv S_I(\omega \approx 0) = eI \coth\frac{\beta e V}{2}\ . \tag{13}$$

In the classical limit $\beta e V \ll 1$ we obtain the Johnson-Nyquist result $S_I \approx 2\beta^{-1} I/V = 2k_B T/R_T$ (this is the thermal noise mentioned above). Here $R_T = 1/G_T = 1/(G_K g) = R_K/g$

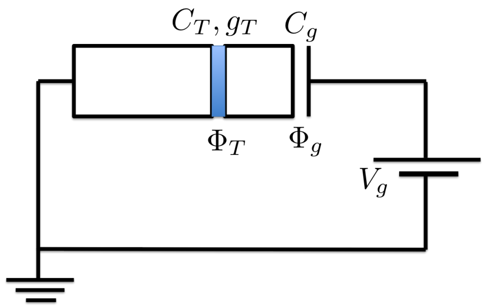

FIG. 1: Single Electron Box. The tunnel junction is characterised by the dimensionless tunnelling conductance g_T and by the capacitance C_T. The island is controlled by the gate voltage V_g via the gate capacitance C_g. To derive the system's Lagrangian and Hamiltonian we introduce the phase drop (dimension of magnetic flux) on the tunnel junction Φ_T and the phase drop on the gate capacitor Φ_g.

is the tunnelling resistance. $R_K = h/e^2$ is the resistance quantum. In the quantum limit $\beta eV \gg 1$ we get the shot noise

$$S_I = eI \ . \tag{14}$$

(In the literature one frequently finds the formulas $S_I = 4k_B T/R_K$ or $S_I = 2eI$. This is because one frequently defines noise correlator as $S(\tau_1, \tau_2) \equiv 2\langle I(\tau_1)I(\tau_2)\rangle - 2\langle I\rangle^2$. This is motivated in part by the fact that in the quantum version one writes $S(\tau_1, \tau_2) \equiv \langle I(\tau_1)I(\tau_2)\rangle + \langle I(\tau_2)I(\tau_1)\rangle - 2\langle I\rangle^2$.)

II. SINGLE ELECTRON BOX

We start by considering the so called single electron box shown in Fig. 1. We introduce the flux (phase) variables. They are related to voltage via $\dot{\Phi} = V$. We first disregard tunnelling and derive the charging Hamiltonian. One way to get the proper Hamiltonian is to take the Coulomb energy and subtract the work performed by the voltage source. Thus, it is actually similar to enthalpy. Instead we derive the Hamiltonian starting from the Lagrangian

$$L = \frac{C_T \dot{\Phi}_T^2}{2} + \frac{C_g \dot{\Phi}_g^2}{2} \ . \tag{15}$$

The sum of all the phases along the loop must vanish and the phase on the voltage source is given by $const. + V_g t$. Thus we obtain (Kirchhoff's law)

$$\dot{\Phi}_g = -\dot{\Phi}_T - V_g \tag{16}$$

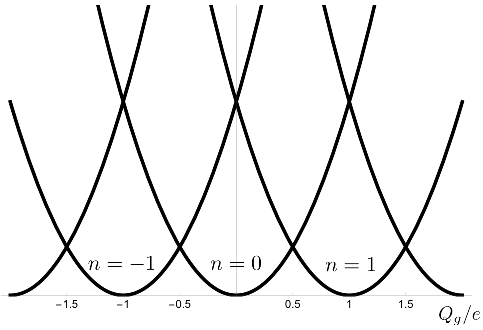

FIG. 2: Coulomb blockade parabolas. Each parabola gives the "energy" at a fixed value of n as a function of the gate charge Q_g.

and the Lagrangian in terms of the only independent generalized coordinate Φ_T reads

$$\mathcal{L} \;=\; \frac{C_T\dot{\Phi}_T^2}{2} + \frac{C_g(\dot{\Phi}_T + V_g)^2}{2} \;. \tag{17}$$

The conjugated momentum (charge) reads

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_T} = C_T\dot{\Phi}_T + C_g(\dot{\Phi}_T + V_g) = (C_T + C_g)\dot{\Phi}_T + C_g V_g \;. \tag{18}$$

Since $C_T\dot{\Phi}_T$ is the charge on the tunnel junction capacitance while $C_g(\dot{\Phi}_T + V_g) = -C_g\dot{\Phi}_g$ is minus the charge on the gate capacitance we conclude that $Q = en$ is the charge on the island. We obtain

$$\dot{\Phi}_T = \frac{Q - C_g V_g}{C_T + C_g} \;. \tag{19}$$

The charging Hamiltonian reads

$$H_C = Q\,\dot{\Phi}_T - \mathcal{L} \;=\; \frac{(Q-Q_g)^2}{2(C_T+C_g)} + const. = \frac{(en-Q_g)^2}{2(C_T+C_g)} + const.\;, \tag{20}$$

where $Q_g \equiv C_g V_g$.

We, thus, obtain the famous Coulomb blockade parabolas (Fig. 2). Each parabola represents the "energy" (value of H_C) at a fixed value of n as a function of the gate charge Q_g.

The island is tunnel coupled to a reservoir. We have to add the tunnelling part to the Hamiltonian. For now we will assume g_T is infinitesimally small and consider the lead as particle reservoir. Thus, the proper potential to consider is $H - \mu_L n$, where μ_L is the (electro)chemical potential of the lead. Here $\mu_L = 0$ since the lead is earthed (we disregard the proper chemical potential (Fermi energy) since we assume it to be the same in the lead

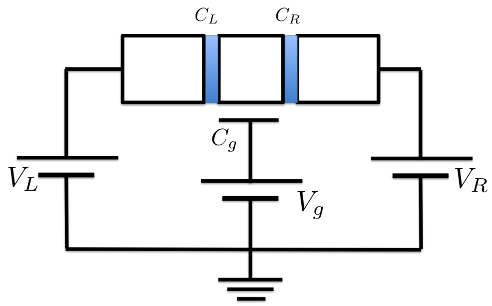

FIG. 3: Single electron transistor.

and in the island). Thus the charge of the island n is determined at $T = 0$ by minimising $H(n)$. From Fig. 2 we observe that, e.g., $n = 0$ corresponds to the minimum of energy for $-0.5 < Q/e < 0.5$. At finite temperature one should use the usual laws of statistical physics, e.g.:

$$\langle n \rangle = \frac{1}{Z} \sum_n n e^{-\beta(H_C(n) - \mu_L n)} \ , \tag{21}$$

where $Z \equiv \sum_n e^{-\beta(H_C(n) - \mu_L n)}$.

III. SINGLE ELECTRON TRANSISTOR

The device is shown in Fig. 3. By Φ_g we denote the phase (in units of flux) dropping across the gate capacitance C_g. The phase dropping across C_L is Φ_L and analogously Φ_R is the phase dropping across C_R. The Kirchhoff's law gives

$$\dot{\Phi}_L + V_L = \dot{\Phi}_R + V_R = \dot{\Phi}_g + V_g = -\dot{\Phi} \ . \tag{22}$$

Here we have introduces the phase Φ (potential $\dot{\Phi}$) of the island. The Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \frac{C_L \dot{\Phi}_L^2}{2} + \frac{C_R \dot{\Phi}_R^2}{2} + \frac{C_g \dot{\Phi}_g^2}{2} \\ &= \frac{C_L(\dot{\Phi} + V_L)^2}{2} + \frac{C_R(\dot{\Phi} + V_R)^2}{2} + \frac{C_g(\dot{\Phi} + V_g)^2}{2} \ . \end{aligned} \tag{23}$$

We calculate again the conjugate to Φ charge

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C_L(\dot{\Phi} + V_L) + C_L(\dot{\Phi} + V_R) + C_g(\dot{\Phi} + V_g) = C_\Sigma \dot{\Phi}_T + Q_g \ , \tag{24}$$

where $C_\Sigma \equiv C_L + C_R + C_g$ and $Q_g \equiv C_L V_L + C_R V_R + C_g V_g$. Again, it is easy to see that Q is the charge of the island, thus $Q = en$. The charging Hamiltonian (enthalpy) reads

$$H_C = Q\,\dot{\Phi} - \mathcal{L} = \frac{(Q - Q_g)^2}{2C_\Sigma} + const. = \frac{(en - Q_g)^2}{2C_\Sigma} + const. \ . \tag{25}$$

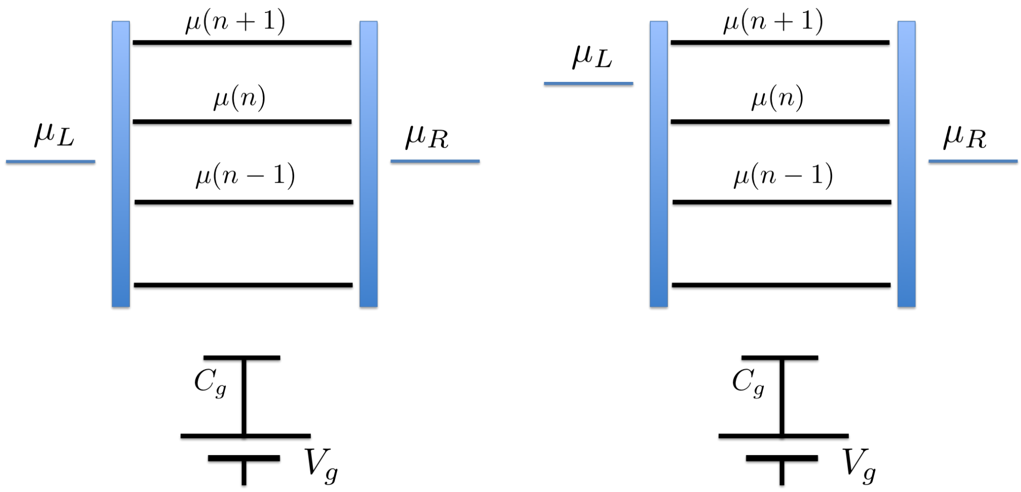

FIG. 4: Staircase of chemical potentials. **Left panel**: Equilibrium. **Right panel**: Non-equilibrium

First, consider the equilibrium situation $V_L = V_R$, i.e., $\mu_L = eV_L = \mu_R = eV_R$. The proper thermodynamic potential is then $G(n) = H_C(n) - \mu_{L/R} n$. At $T = 0$ we just have to minimise $H_C(n) - \mu_L n$ in order to determine the charge of the island. Since n is discrete, the minimum is achieved if $G(n+1) > G(n)$ and $G(n-1) > G(n)$. This gives

$$H_C(n+1) - H_C(n) > \mu_{L/R} \quad \text{and} \quad H_C(n) - H_C(n-1) < \mu_{L/R} \ . \tag{26}$$

It is convenient to introduce the chemical potential of the island as

$$\mu(n) \equiv H_C(n+1) - H_C(n) \ . \tag{27}$$

Then the $T = 0$ equilibrium condition reads:

$$\mu(n) > \mu_{L/R} > \mu(n-1) \ . \tag{28}$$

It is easy to calculate $\mu(n)$:

$$\begin{aligned} \mu(n) &= \frac{(e(n+1) - Q_g)^2}{2C_\Sigma} - \frac{(en - Q_g)^2}{2C_\Sigma} = \frac{e^2 + 2e(en - Q_g)}{2C_\Sigma} \\ &= \frac{e^2}{C_\Sigma}\left[\frac{1}{2} + n - \frac{Q_g}{e}\right] \ . \end{aligned} \tag{29}$$

We thus obtain a staircase as indicated in Fig. 4 (left panel). When a voltage is applied the situation changes as shown in Fig. 4 (right panel).

A. Transport in SET

Fig. 4 helps to formulate the general master equation describing transport in an SET. We introduce the probability $p(n)$ that the charge of the island equals n. Note the difference with the tunnel junction, where $p(n)$ denoted the probability that n charges have tunnelled. The tunnelling rates depend now on the state of the island. Analogously to Eq. (2) we obtain the rate of tunnelling from the left lead to the island given the island was before in the state with n charges.

$$\Gamma_{L\to I}(n) = \frac{g_L}{h}\frac{\Delta\mu_L(n)}{1-e^{-\beta\Delta\mu_L(n)}} \ , \tag{30}$$

where $\Delta\mu_L(n) \equiv \mu_L - \mu(n)$. Here $g_R = 2\times(2\pi)^2|t_L|^2\rho_L\rho_I$ is the dimensionless tunnelling conductance of the left junction. Similarly

$$\Gamma_{R\to I}(n) = \frac{g_R}{h}\frac{\Delta\mu_R(n)}{1-e^{-\beta\Delta\mu_R(n)}} \ , \tag{31}$$

where $\Delta\mu_R(n) \equiv \mu_R - \mu(n)$. The rates of tunnelling from the island are given by (recall that n here means the charge before the tunnelling)

$$\Gamma_{I\to L}(n) = \frac{g_L}{h}\frac{[-\Delta\mu_L(n-1)]}{1-e^{\beta\Delta\mu_L(n-1)}} \ , \tag{32}$$

and

$$\Gamma_{I\to R}(n) = \frac{g_R}{h}\frac{[-\Delta\mu_R(n-1)]}{1-e^{\beta\Delta\mu_R(n-1)}} \ . \tag{33}$$

The master equation reads

$$\begin{aligned} \dot{p}(n) &= [\Gamma_{L\to I}(n-1)+\Gamma_{R\to I}(n-1)]\,p(n-1) \\ &+ [\Gamma_{I\to L}(n+1)+\Gamma_{I\to R}(n+1)]\,p(n+1) \\ &- [\Gamma_{L\to I}(n)+\Gamma_{R\to I}(n)+\Gamma_{I\to L}(n)+\Gamma_{I\to R}(n)]\,p(n) \ . \end{aligned} \tag{34}$$

This master equation allows to find the steady state occupation of the island $\langle n\rangle$, but we need more to determine current and noise. The method of full counting statistics helps again. We define the probability $p(n,m)$ that the island has n charges and m charges have tunnelled through the right junction (this choice is arbitrary). Then, for the Fourier transform $p(n,k) \equiv \sum_m p(n,m)e^{-ikm}$ we obtain

$$\begin{aligned} \dot{p}(n,k) &= \left[\Gamma_{L\to I}(n-1)+e^{ik}\Gamma_{R\to I}(n-1)\right]p(n-1,k) \\ &+ \left[\Gamma_{I\to L}(n+1)+e^{-ik}\Gamma_{I\to R}(n+1)\right]p(n+1,k) \\ &- [\Gamma_{L\to I}(n)+\Gamma_{R\to I}(n)+\Gamma_{I\to L}(n)+\Gamma_{I\to R}(n)]\,p(n,k) \ . \end{aligned} \tag{35}$$

The generating function is then obtained as

$$\chi(k) = \sum_n p(n, k) \ . \tag{36}$$

We will not analyse in detail all the regimes. Qualitative description will be given in the lecture.