The General Theory Of Relativity In Five Pages

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1 Introduction

A translation of Newton's first law of motion (1687) reads: "Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed". Incidentally, a uniform straight motion in space is just a straight motion in spacetime. We can read it backwards by asking, what is a straight line in spacetime? Take a particle, let it go freely, and see.

Since Galileo, it is known that gravity is universal, which is also stated as an equivalence principle: "The trajectory of a point mass in a gravitational field depends only on its initial position and velocity, and is independent of its composition and structure".

Putting those two together may lead one to conclude that gravity is not a force, but a manifestation of spacetime upon all particles propagate, and a free falling particle is traveling on a "straight line" of that spacetime.

The Newtonian equation of motion for a particle in a gravitational field is

$$\boldsymbol{a} = -\nabla\phi \tag{1}$$

Where $\boldsymbol{a}$ is the three-acceleration and ϕ is the scalar gravitational potential field. The field equation for ϕ is the Poisson's equation

$$\nabla^2\phi = 4\pi\rho \tag{2}$$

Where ρ is a mass density, the source of the gravitational field.

Equation (1) clearly does not describe a straight line in three Euclidean space, because of the gravity "force". But, Cartan (1923,1924) showed that indeed it is possible to formulate Newtonian gravity as a geometrical object.

A geometrical pedagogical regression is in order.

There is a freedom to choose a basis for each vector tangent space at each point of the spacetime manifold. More abstractly, like in gauge field theories in particle physics, this space can be a representation space of some Lie algebra like $u(n)$. Borrowing the terminology, this choice of basis is called a *gauge*. A coordinate basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$ is one that is induced from the coordinates on a chart of spacetime (pointing in their direction). Thus we will use synonymously the terms "coordinates" and "gauge". A *connection* on a manifold is an assignment to identify parallel vectors in different tangent spaces, which allows a covariant differentiation of vector fields. In a coordinate basis the *covariant derivative* of a vector field $V = V^\mu \frac{\partial}{\partial x^\mu}$ in the direction $\frac{\partial}{\partial x^\nu}$ is

$$(\nabla_\nu V)^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho} V^\rho \tag{3}$$

How the basis of each tangent space ("frame") changes from point to point is the connection coefficients. In coordinate basis

$$\Gamma^\rho_{\mu\nu} := dx^\rho \left(\nabla_\mu \frac{\partial}{\partial x^\nu}\right) \tag{4}$$

$\Gamma^\rho_{\mu\nu}$ is the ρ-th vector component of the covariant derivative in the μ direction of the base vector $\frac{\partial}{\partial x^\nu}$.

The covariant derivative quantifies the rate of change of a tensor field in comparison to what the tensor would be if it were "parallel transported". In a curved space, the result of parallel transporting a vector from one point to another will depend on the path taken between the points. For a torsion free connection, parallel transporting a vector field V in an infinitesimal loop formed by vectors X, Y, results in a new vector, depending only on the curvature at the point, manifested in the linear transformation of the vector by the *Riemann curvature tensor* R

$$R(X,Y)\, V = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) V \tag{5}$$

In shorter terms, the Riemann curvature tensor is the covariant exterior derivative of the connection $R = d\Gamma + \Gamma \wedge \Gamma$, which in coordinates reads

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{6}$$

Going back to Cartan, in this view the Newtonian spacetime has an absolute time function t, which defines a constant t Euclidean space slices, and a connection ∇ on spacetime. *Galilean coordinates* are defined such that $x^0 = t$, the spatial metric is δ_{ij}, and the only non vanishing connection coefficients are $\Gamma^i_{00} = \partial^i \phi$. So space is flat ($R^l_{ijk} = 0$) but spacetime is curved, and the gravitational potential can be encoded in a connection with one degree of freedom ϕ. Without gravity ($\phi = const$) the Galilean coordinates are *inertial coordinates*, where all $\Gamma^\rho_{\mu\nu} = 0$ and spacetime is flat $R^\rho_{\sigma\mu\nu} = 0$. In the following sections we will see this approach reproduces equations (1) and (2).

However, Maxwell's equations for the electromagnetic field (1865) suggest all observers measure the same speed of light, which led Einstein in the theory of special relativity (1905) to conclude that spacetime has a different structure. It is endowed with a Lorentzian metric, which defines a light cone causal structure in the tangent space. Newtonian gravity does not respect causality since (2) has no time derivatives, which suggest an instantaneous response ("action at a distance"), faster then the speed of light. Einstein's general relativity (1915) is Einstein's theory of relativistic gravity, where the gravitational potential field is a dynamical Lorentzian metric g, of a curved spacetime. We see that a theory of gravity as the geometry of spacetime is induced from the kind of matter that propagate on it.

2 Equations of Motion and Actions

2.1 Point Particle Motion

2.1.1 Equation of motion

Combining Newton's first law and the equivalence principle, the equation of motion for a free falling particle needs to express that if at each moment there is no force acting, then at each moment the particle is moving in a straight line in spacetime.

Let there be a curve γ, parameterized by λ. Let us denote by v_γ the tangent vector to this curve at some point $\gamma(\lambda)$, which is the velocity vector. The directional covariant derivative along the curve at this point is the covariant derivative in the direction of v_γ, $\frac{D}{d\lambda} := \nabla_{v_\gamma}$.

A tensor field T is said to be *parallel transported* along the curve γ if at each point of the curve its directional covariant derivative vanishes $\frac{D}{d\lambda}T = 0$. This is just the mathematical covariant formulation of the requirement for the tensor to "be kept constant" along the curve.

In flat space, a straight line is a curve along which its tangent vector is being parallel transported. Define the covariant spacetime acceleration vector by $a_\gamma := \frac{D}{d\lambda}v_\gamma$, then one generalization of a straight line to a curved manifold, called an *auto-parallel* curve, is the solution of the equation

$$a_\gamma = 0 \tag{7}$$

Equation (7) is the geometrical formulation of Newton's first law, for arbitrary coordinates ("fictitious forces arising in non-inertial frame") and curved spacetime (gravity).

In a chart with coordinates x^μ, the components of this differential vector equation takes the form

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{8}$$

Indeed, choosing the coordinate time t to be the curve parameter and Galilean coordinates, the spatial components of equation (8) is exactly Newton's second law with only a gravitational force (1).

2.1.2 Action

A relativistic spacetime has a Lorentzian metric g. The *metric* tensor field defines an inner product in each tangent space, so it is a symmetric $(0,2)$ tensor, where in coordinate basis $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$. The norm (squared) of a vector field with infinitesimal components dx^μ is the invariant infinitesimal line element (squared)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{9}$$

Since the metric measure lengths, we may look at another generalization of a straight line, namely a curve between two points with extremal length, called a *geodesic*. Geodesics are the stationary functions of the length functional

$$L = \int_a^b d\lambda \sqrt{g\left(v_\gamma, v_\gamma\right)} \tag{10}$$

For a massive particle, in the language of Lagrangian mechanics, the *relativistic free point-particle action* is

$$S_{pp} = -m \int d\tau \tag{11}$$

Where the line integral is along the particle worldline. This coupling of the particle to gravity is called *minimal*, it is solely though the infinitesimal proper time length at each point $\gamma\left(\lambda\right)$, $d\tau = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$. The Euler-Lagrange equation of this action is called the *geodesic equation*.

Einstein's equivalence principle demands that in a local region there is no "gravity" that cannot be gauged away. Geometrically speaking, that at every point spacetime has a *local Lorentz frame* (at a point $g_{\mu\nu} = \eta_{\mu\nu} = diag\left(1, -1, -1, -1\right)$ the Minkowski metric, first derivative vanish, second derivative does not). In a Lorentz frame an extremal path has the form $\frac{d^2 x^\mu}{d\tau^2} = 0$, which is identical to an Auto-parallel in a region with no curvature (gravity) in inertial coordinates (all $\Gamma^\rho_{\mu\nu} = 0$). Thus geodesics and Auto-parallels coincide, and equation (8) **is** the geodesic equation (with an affine parameter) where the connection coefficients are completely determined by the metric by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{12}$$

It is equivalent to a demand that ∇ is a Levi-Civita connection, i.e., a metric compatibility condition that the metric is covariantly constant $\nabla g = 0$, and a torsion free condition $\Gamma^\rho_{[\mu\nu]} = 0$.

The physical conclusion is that although the field strength is the curvature tensor, the gravitational field potential in GR is not a scalar field like in Newtonian gravity, nor the connection (compare to $u(n)$ gauge theories where the connection A is the gauge field), but the metric g, and all arise from it. Thus, the curvature encapsulates real tidal forces (first derivative of the force, second derivative of the potential), but the gravitational "force" itself (the connection, first derivative of the potential) can be gauged away in a local region, unlike any other kind of force, which is exactly Einstein's equivalence principle.

2.2 Gravitational Field Motion

2.2.1 Equation of motion

Any kind of energy (like mass, kinetic, electromagnetic) is a source of the gravitational field $g_{\mu\nu}(x)$, and is encoded in an energy-momentum tensor $T_{\mu\nu}$. The equation of motion for the gravitational field is the *Einstein field equation* (with no cosmological constant)

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{13}$$

Where $R_{\mu\nu}$ is the *Ricci tensor*, a contraction of the Riemann tensor $R_{\mu\nu} := R^{\rho}_{\mu\rho\nu}$, R is the *Ricci scalar*, the trace of the Ricci tensor $R := g^{\mu\nu} R_{\mu\nu}$, and $G_{\mu\nu}$ is called the *Einstein tensor*. In Newtonian gravity in Galilean coordinates the only non-vanishing Ricci tensor coefficient is $R_{00} = \nabla^2 \phi$. Comparing to Poisson's equation (2) suggest the field equation $R_{\mu\nu} = 4\pi T_{\mu\nu}$, where $T_{00} = \rho$. It took Einstein some time to find the second term on the curvature side of (13), yet it is necessary for local energy-momentum conservation to hold $divT = 0$. Unlike the Poisson equation for Newtonian gravity, or equation of motion for the electromagnetic field, the Einstein equation is non linear in the metric field i.e, the gravitational field is self interacting. There are only few known exact solutions, with assumed symmetries, most famous are the vacuum spherical symmetric solution called the Schwarzschild metric which also suggest the existence of black holes, and cosmological descriptions of the universe. Perturbations around an exact solution can be made, for instance, a linear tensor perturbation around flat spacetime describes a gravitational wave.

2.2.2 Action

Equation (13) in vacuum ($T_{\mu\nu} = 0$) can be derived from the *Einstein-Hilbert action*

$$S_{EH} = \frac{1}{16\pi} \int \sqrt{|g|} d^d x R \tag{14}$$

Where $\sqrt{|g|} d^d x$ is the invariant volume form and g is the determinant of the metric.