Chapter 3
Single Particle Motion in Electric and Magnetic Fields

'Twas brillig, and the slithy toves
Did gyre and gimble in the wabe.

Lewis Carroll, Jabberwocky

Plasmas belong to two different categories, unmagnetized and magnetized. The plasma in a fluorescent tube is unmagnetized, because the motion of electrons and ions is determined by electric fields and collisions, and the Earth magnetic field is too weak to bend the trajectories. The ionosphere, the magnetosphere, the solar wind, the interstellar medium and the solar surface are examples for natural magnetized plasmas. There, the motion of the particles is strongly affected by the magnetic field.

This chapter is focused on the motion of individual charged particles in given electric and magnetic fields. Of particular importance is the quest for magnetic confinement of plasmas. The inhomogeneity and curvature of magnetic field lines, or the variation of the fields in time cause complex particle motion. The model of *single particle motion* neglects the influence of particle currents on the electric and magnetic fields. In this respect, the model is still incomplete. Nevertheless, from an understanding of particle motion the reader will gain insight into the basic properties of a plasma that is subjected to electromagnetic fields.

3.1 Motion in Static Electric and Magnetic Fields

3.1.1 Basic Equations

The starting point for establishing the single-particle model is Newton's equation[1] for the motion of a particle of mass m and charge q in a given electric field **E** and magnetic field **B**

[1] Sometimes called Newton–Lorentz equation.

A. Piel, *Plasma Physics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-63427-2_3

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ , \tag{3.1}$$

in which the dot represents the time derivative at the position of the particle. This equation can be solved in rigid mathematical terms only for simple cases, e.g., homogeneous and stationary fields.

3.1.2 Cyclotron Frequencies

Let us first consider the case of a homogeneous and stationary magnetic field $\mathbf{B} = (0, 0, B_z)$ and a vanishing electric field $\mathbf{E} = 0$. The magnetic field is chosen as z-axis because of the cylindrical symmetry about the B-field direction. Then, we obtain Newton's equation of motion in cartesian coordinates as

$$\begin{aligned} \dot{v}_x &= +v_y \frac{q}{m} B_z \\ \dot{v}_y &= -v_x \frac{q}{m} B_z \\ \dot{v}_z &= 0 \ . \end{aligned} \tag{3.2}$$

By combining the equations for the x and y-motion we obtain the differential equation for a harmonic oscillator

$$\ddot{v}_{x,y} = -\left(\frac{qB_z}{m}\right)^2 v_{x,y} \ . \tag{3.3}$$

This harmonic oscillator describes a periodic motion at a frequency

$$\omega_{\mathrm{c}} = \frac{|q|}{m} B_z \ , \tag{3.4}$$

which we call the *cyclotron frequency*. Inserting numbers for q, B, and m we find the cyclotron frequency of an electron in a magnetic field of 1 T at $\omega_{\mathrm{ce}} = 1.759 \times 10^{11}\,\mathrm{s}^{-1} = 2\pi \times 27.99$ GHz. At the same magnetic field, the proton cyclotron frequency is $\omega_{\mathrm{cp}} = 9.579 \times 10^7\,\mathrm{s}^{-1} = 2\pi \times 15.25$ MHz.

In the x–y plane, a particle with perpendicular velocity $v_\perp$ performs a circular orbit with the gyroradius or Larmor radius, named after the Irish physicist Joseph Larmor (1857–1942),

$$r_{\mathrm{L}} = \frac{v_\perp}{\omega_{\mathrm{c}}} \ . \tag{3.5}$$

When the initial velocity v_z along the magnetic field is nonzero, the orbit becomes a helix of constant pitch about the magnetic field direction (z). The motion about a magnetic field line is referred to as gyromotion or gyroorbit.

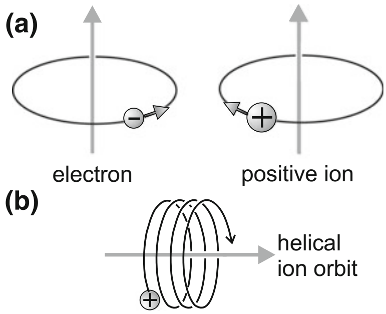

Fig. 3.1 **a** Gyromotion of electrons and ions. Note that electrons perform a right-handed motion about the magnetic field (Consider the thumb of your right hand representing the magnetic field direction, then the fingers give the sense of electron motion). **b** Helical orbit of an ion

The sense of rotation about the magnetic field depends on the sign of the particle's charge. Electrons move in a *right-handed*, positive ions in a *left-handed* orbit (see Fig. 3.1).

When the ions (electrons) can perform complete gyroorbits, the ions (electrons) are called *magnetized*. This is the case, when the gyroorbits are not interrupted by collisions. A condition for this to happen is that the ion (electron) collision frequency (cf. Sect. 4.2.2) is smaller than the ion (electron) cyclotron frequency. Usually, this condition can be better fulfilled by electrons than by ions. A more detailed discussion of the interplay of gyromotion and collisions can be found in Sect. 4.3.4.

The ions in a gas discharge plasma in the presence of the Earth's magnetic field can be considered as unmagnetized. This may be unrelated to the frequency of collisions but is rather a consequence of the size of the gyroradius, which is larger than the diameter of the discharge tube. Then, this segment of the gyroorbit can be considered as sufficiently straight—as in the unmagnetized case.

3.1.3 The Earth Magnetic Field

In the immediate neighborhood of the Earth, the magnetic field has the shape of a dipole field (Fig. 3.2). The source of this field can be represented by a magnetic dipole at the Earth's center with a magnetic moment of $|\mathbf{M}| = 7.3 \times 10^{22}\,\mathrm{A\,m^2}$. This dipole is tilted from the axis of rotation leading to a deviation of the magnetic poles from the geographic poles. The Earth magnetic field is generated by electric currents in the Earth's core. The general shape of the distorted Earth magnetic field under the influence of the solar wind was shown in Fig. 1.6.

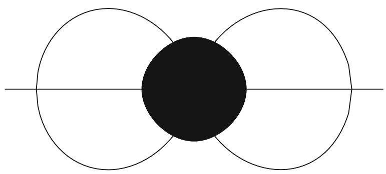

Fig. 3.2 The unperturbed dipole field near the Earth. The *horizontal line* marks the (magnetic) equatorial plane

The magnetic induction $\mathbf{B}(\mathbf{r})$ of a dipole field is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3\mathbf{r}(\mathbf{r} \cdot \mathbf{M}) - r^2\mathbf{M}}{r^5} . \tag{3.6}$$

Here $\mathbf{r}$ is a vector pointing from the magnetic dipole to the field point. At the author's location (54.3° N.; 10.1° E), the Earth magnetic field has a horizontal component $B_\mathrm{h} = 17{,}700\,\mathrm{nT}$ and a vertical component $B_\mathrm{v} = -46{,}150\,\mathrm{nT}$.

3.1.4 E × B Drift

When we now allow for a stationary and homogeneous electric field, we can choose the orientation of our coordinate system, without loss of generality, to have electric and magnetic field in the x–z plane, $\mathbf{E} = (E_x, 0, E_z)$ and $\mathbf{B} = (0, 0, B_z)$. In this case, Newton's equation of motion reads

$$\begin{aligned} \dot{v}_x &= \frac{q}{m}\left(E_x + v_y B_z\right) \\ \dot{v}_y &= \frac{q}{m}(-v_x B_z) \\ \dot{v}_z &= \frac{q}{m} E_z . \end{aligned} \tag{3.7}$$

The motion along the magnetic field is now accelerated but independent of the motion in the x–y plane. According to the principle of superposition of motions, we can consider both effects separately. For the x–y plane, the motion can again be decomposed

$$\begin{aligned} \ddot{v}_x &= -\omega_\mathrm{c}^2 v_x \\ \ddot{v}_y &= -\omega_\mathrm{c}^2 (v_y + E_x/B_z) . \end{aligned} \tag{3.8}$$

Again, we find a harmonic oscillation in x-direction, but the motion in y-direction is more complex. In a moving frame of reference, $\tilde{v}_y = v_y + E_x/B_z$, which moves at a constant velocity $-E_x/B_z$ in negative y-direction, we obtain a simple harmonic motion

$$\ddot{\tilde{v}}_y = -\omega_c^2 \tilde{v}_y . \tag{3.9}$$

Thus the solution for the velocities is the superposition of a circular orbit and a constant motion in the same plane. This constant motion is called the E × B-drift. For a particle that is initially at rest, the solution reads

$$\begin{aligned} v_x &= \frac{E_x}{B_z} \sin \omega_\mathrm{c} t \\ v_y &= \frac{E_x}{B_z} \left[\cos \omega_\mathrm{c} t - 1\right] . \end{aligned} \tag{3.10}$$

A typical trajectory is shown in Fig. 3.3. Mathematically, the trajectory is a cycloid. For a positive particle starting at $t = 0$ from the origin, the trajectory is described by

$$x = \frac{E_x}{B_z \omega_c} [1 - \cos(\omega_c t)] \ , \quad y = \frac{E_x}{B_z \omega_c} [\sin(\omega_c t) - \omega_c t] \tag{3.11}$$

The E × B-drift can also be understood from energy considerations. On the high-potential side, the kinetic energy is small, which makes the instantaneous gyroradius small. On the low-potential side, the ion has gained kinetic energy from the electric field, which makes the gyro-radius larger. The combination of these two effects results in a cycloidal motion.

It is a peculiarity of the E × B-drift that negative electrons and positive ions experience the same sign of the drift velocity. This is a consequence of the fact that the applied electric field force $q\mathbf{E}$ and the resulting Lorentz force $q\mathbf{v} \times \mathbf{B}$ both depend on the sign of q, which cancels in the result. This effect can also be seen in Fig. 3.3.

In vector notation, the E × B drift velocity is given by

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \ . \tag{3.12}$$

3.1.5 Gravitational Drift

When we consider the ionospheric plasma at the magnetic equator, we find a similar situation to crossed electric and magnetic fields. Here, the force of gravity, $m\mathbf{g}$, is perpendicular to the (horizontal) magnetic field lines. Neglecting collisions, which

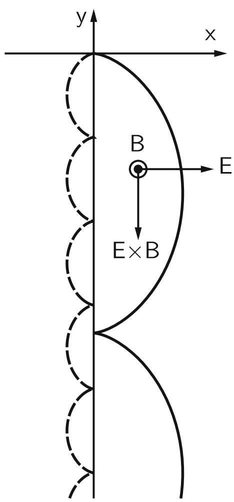

Fig. 3.3 Cycloidal trajectory resulting from the superposition of gyro-motion and E × B-drift. The electric field is oriented along the x-axis, the magnetic field is perpendicular to the x–y plane. Note that ions (*full line*) and electrons (*dashed line*) have the same drift direction. An artificial electron mass, $m_e = 0.3\, m_{ion}$, was assumed here for clarity

indeed are important in the lower ionosphere, Newton's equation of motion

$$m\dot{\mathbf{v}} = m\mathbf{g} + q\mathbf{v} \times \mathbf{B} \tag{3.13}$$

can be translated into the case of E × B motion by introducing an *equivalent electric field* $\mathbf{E} = (m/q)\mathbf{g}$. Without solving (3.13) we can immediately give the result for the velocity of the gravitational drift

$$\mathbf{v}_{\mathrm{g}} = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \; . \tag{3.14}$$

Note that now the drift velocity depends on mass and charge. In particular, electrons and positive ions will drift in opposite directions. The gravitational drift is responsible for an equatorial net electric current that is driven by the weight force on the plasma. However, the collisionless approximation is too crude to give its correct magnitude.

3.1.6 Application: Confinement of Nonneutral Plasmas

A nonneutral plasma consists of only one sort of charged particles, often positive ions. Figure 3.4 shows a typical magnetic trap of the Penning–Malmberg type, which is suitable for trapping electrons or ions. These traps use strong magnetic fields of $|B| > 1\,\mathrm{T}$. A review of experiments with this device can be found in [59].

The axial magnetic field B provides magnetic confinement by having the ions gyrate about the magnetic field line. The axial confinement is achieved by electric fields from the positively-biased outer cylinders that repel the ions towards the center. The ion cloud represents a region of positive space charge. From Poisson's equation in cylindrical geometry

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) = \frac{n_{\mathrm{i}}e}{\varepsilon_0} \tag{3.15}$$

we obtain $E_r = \frac{1}{2}n_{\mathrm{i}}er\varepsilon_0^{-1}$, i.e., the electric field increases linearly from the center to the edge of the ion cloud. Hence, the E × B velocity increases in the same man-

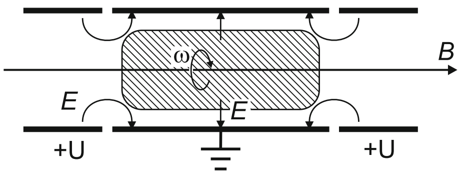

Fig. 3.4 The Penning–Malmberg trap for confining a nonneutral plasma of positive ions (hatched area) uses three cylindrical tube electrodes aligned with a strong magnetic field B

ner, which means that the cloud rotates as a rigid body with an angular frequency $\omega = E(rB)^{-1} = \frac{1}{2} n_i e (\varepsilon_0 B)^{-1}$. Ions can be cooled to milli-Kelvin temperatures by a technique called laser-doppler cooling [60]. Such devices can be used to trap antiprotons for a sufficiently long time to recombine with positrons from a radioactive source to form antihydrogen [61–63].

3.2 The Drift Approximation

In this section, approximate solutions are sought for the case of inhomogeneous and curved magnetic fields. We will discuss the influence of inhomogeneity and curvature in separate steps although these two aspects are intertwined by Maxwell's equations.

3.2.1 The Concept of a Guiding Center

We have seen that the effect of an external force on a gyrating particle can be described as a net *drift motion* that is superimposed on the gyromotion. We will apply this idea to the motion in inhomogeneous magnetic fields. For this purpose, we assume that the true particle orbit can be decomposed into a circular orbit about a local *guiding center* and a drift motion of the guiding center (see Fig. 3.5). For the drift motion we calculate a net force, which is the average over one gyro-period. This net force is then converted into an equivalent electric field—as in the case of the gravitational drift—and the drift velocity is obtained from (3.12).

Such an approximation requires that the gradient of the magnetic field is small. This can be expressed by the requirement that the change of the magnetic field across one gyroradius is small compared to the magnitude of the magnetic field at the guiding center

$$r_L \frac{\partial B_z}{\partial r} \ll B_z \; . \tag{3.16}$$

The guiding center approximation is in fact more than a simple Taylor expansion of the fields. The resulting expressions have a wider range of applicability than expected from the requirement of (3.16).

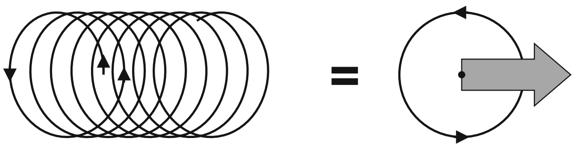

Fig. 3.5 The concept of a guiding center decomposes the actual cycloidal orbit into a circular motion about the guiding center and a drift motion of the guiding center

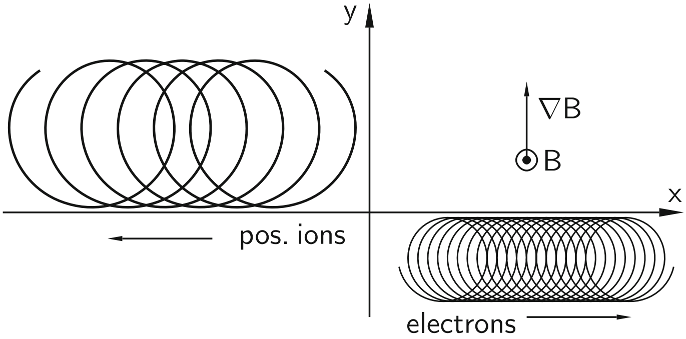

Fig. 3.6 Gradient drift of electrons and ions. This drift is charge sensitive. Note that the instantaneous curvature of the trajectory is smaller in regions of stronger magnetic field

3.2.2 Gradient Drift

In a first step, we assume that the magnetic field is inhomogeneous, but that the field lines are straight and parallel. The influence of field line curvature will be discussed separately below. The particle experiences a Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. In the geometry shown in Fig. 3.6, the y-component of this force is given by

$$F_y = -qv_x B_z(y) \; , \tag{3.17}$$

where $B_z(y)$ is the true magnetic field at the position of the particle. This can be estimated from the field at the guiding center by Taylor expansion, $B_z(y) = B_0 + y(t)(\partial B_z/\partial y)$, yielding

$$F_y = -qv_\perp \sin(\omega_c t)\left[B_0 \pm r_L \sin(\omega_c t)\frac{\partial B_z}{\partial y}\right] . \tag{3.18}$$

$v_\perp$ is the orbit velocity of the gyrating particle in a plane perpendicular to the magnetic field. The upper sign in this expression corresponds to positive, the lower sign to negative particles. When we put B_0 outside the brackets, the small expansion parameter becomes visible

$$F_y = -qv_\perp \sin(\omega_c t) B_0 \left[1 \pm \frac{r_L(\partial B_z/\partial y)}{B_0}\sin(\omega_c t)\right] . \tag{3.19}$$

According to the recipe given above, we now need to average the force over one gyroperiod and make use of the fact that the average of a sine function over one period is zero whereas the average of the sine-square is 1/2, giving

$$\langle F_y \rangle = \mp q v_\perp r_L \frac{\partial B_z}{\partial y}\langle \sin^2(\omega_c t)\rangle = e v_\perp r_L \frac{\partial B_z}{\partial y}\frac{1}{2} . \tag{3.20}$$

The resulting average force is independent of the sign of the charge. However, the corresponding equivalent electric field $E = (1/q)\langle F_y \rangle$ is charge sensitive. Hence, we obtain a drift velocity

$$\mathbf{v}_{\nabla B} = \frac{1}{q} \frac{\langle \mathbf{F} \rangle \times \mathbf{B}}{B^2} = \pm \frac{1}{2} v_\perp r_L \frac{\mathbf{B} \times \nabla |\mathbf{B}|}{B^2} \; . \tag{3.21}$$

This is the velocity of the *gradient drift*. The charge dependence leads to charge separation and to the formation of a net current across the magnetic field.

3.2.3 Curvature Drift

In the second step, we now consider curved field lines with a constant radius of curvature R_c. At the same time, we neglect the gradient of the magnetic field, which we have already discussed in the preceding paragraph. The curvature drift is an effect of motion along the field line, where the particle experiences a centrifugal force F_c from the curvature

$$\langle \mathbf{F}_c \rangle \;=\; \frac{m v_z^2}{R_c} \mathbf{e}_R \; . \tag{3.22}$$

Here, v_z is the parallel velocity and $\mathbf{e}_R$ the unit vector in radial direction. We have retained the average over a gyroperiod for compatibility with calculations above. This reflects the idea that the particle experiences a constant net curvature during one gyroorbit. This force leads to a drift velocity

$$\mathbf{v}_R = \frac{1}{q} \frac{\mathbf{F}_c \times \mathbf{B}}{B^2} = \frac{m v_z^2}{q B^2} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2} \; . \tag{3.23}$$

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3.3 The Magnetic Mirror

In this section we consider a situation where the gradient of the magnetic field is parallel to the field direction (see Fig. 3.8). For mathematical simplicity we assume a bundle of straight field lines with rotational symmetry about a central field line. The guiding center is assumed to move along this central field line. We are allowed to neglect the curvature of the field lines, which would lead to a curvature drift as discussed above.

3.3.1 Longitudinal Gradient

In such a type of inhomogeneous magnetic field, a charged particle experiences a constant net Lorentz force $q(\mathbf{v}_\perp \times \mathbf{B}_r)$ that has its origin in the radial component of the magnetic field. The force vector is oriented along the central field line and points in the direction of a weaker field. Note that the sign of the charge cancels because of the reversed sense of gyration for negative particles. This force acts to decelerate and eventually reflect a particle that has originally moved into the region of stronger field. Therefore, this field geometry is called a *magnetic mirror*.

We obtain the radial part of the magnetic field from the vanishing of the divergence of the magnetic induction

$$0 = \nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial}{\partial z} B_z \ . \tag{3.30}$$

When we prescribe the longitudinal gradient $\partial B_z/\partial z$ at $r = 0$, and assume this as approximately constant, we can integrate (3.30)

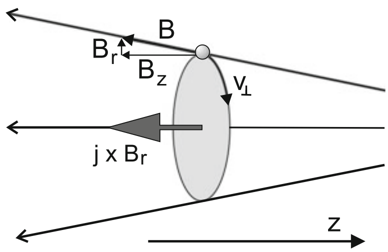

Fig. 3.8 The magnetic field lines with a longitudinal gradient form a magnetic mirror

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}\mathrm{d}r \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \tag{3.31}$$

and obtain the radial magnetic field as

$$B_r \approx -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \quad . \tag{3.32}$$

Then, the net Lorentz force in z-direction, acting on the ring current with a gyroradius r_L, is

$$\langle F_z\rangle = -\frac{1}{2}qv_\perp r_\mathrm{L}\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \quad . \tag{3.33}$$

This force leads to an accelerated motion of the guiding center along the magnetic field. Hence, the case of a longitudinal gradient does *not* lead to a new drift velocity. Drift motion is only found when the averaged force is perpendicular to the magnetic field direction.

3.3.2 *Magnetic Moment*

The circular orbit of the particle about the central field line in the geometry of Fig. 3.8 can be considered as an electric current. This ring current has an associated magnetic moment $\boldsymbol{\mu}$, which is the product of the current I flowing at the edge of a circular disk with Larmor radius r_L and the area A of this disk. The current is given by the charge q performing a revolution in the gyroperiod T_c:

$$|\boldsymbol{\mu}| = IA = \frac{|q|}{T_\mathrm{c}}\pi r_\mathrm{L}^2 = |q|\frac{\omega_\mathrm{c}}{2\pi}\pi\left(\frac{v_\perp}{\omega_\mathrm{c}}\right)^2 = \frac{mv_\perp^2}{2B} = \frac{W_\perp}{B} \quad . \tag{3.34}$$

The magnetic moment is a vector that is antiparallel to the ambient magnetic field B (Fig. 3.9) and—according to Lenz's rule—weakens the external field. With this definition of the magnetic moment, we can rewrite (3.33)

$$\langle F_z\rangle = -\mu\frac{\partial B_z}{\partial z} \tag{3.35}$$

with $\mu = |\boldsymbol{\mu}|$. This shows that the gyrating particle experiences a force like a piece of diamagnetic matter in an inhomogeneous magnetic field. The diamagnetism results from the left-handed motion of a positive ion, which creates a magnetic dipole that is antiparallel to the acting magnetic field. The same is true for electrons, which have the opposite charge and the opposite sense of gyration.

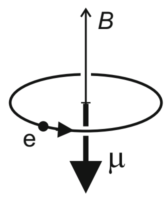

Fig. 3.9 The magnetic moment $\boldsymbol{\mu}$ of a gyroorbit is antiparallel to the magnetic field. This makes a magnetized plasma diamagnetic

3.4 Adiabatic Invariants

It is shown in classical mechanics that the action integral over a periodic orbit, $\oint p\,\mathrm{d}q$, is a conserved quantity of the system. This concept can be extended to weak gradients, in which the orbit is nearly periodic. The associated action integrals then are no longer strict invariants but become *adiabatic invariants*.

3.4.1 The Magnetic Moment as First Invariant

When we assume that the diamagnetic force in (3.35) is a valid description, we obtain an energy relation for the motion of the guiding center by multiplying with v_z

$$\begin{aligned} m\dot{v}_z &= -\mu\frac{\partial B}{\partial z} \\ m v_z \dot{v}_z &= -\mu\frac{\partial B}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m v_z^2\right) &= -\mu\frac{\mathrm{d}B}{\mathrm{d}t}\ . \end{aligned} \tag{3.36}$$

Here, $\mathrm{d}B/\mathrm{d}t$ is the change in the magnetic field, which the guiding center experiences by moving along the central field line. For the guiding center, we have no radial magnetic field and therefore can drop the index z. A time-invariant magnetic field does not alter the kinetic energy, as can be seen from $\mathbf{F}\cdot\mathrm{d}\mathbf{s} = q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,\mathrm{d}t = 0$, because the Lorentz force is always perpendicular to the trajectory. The change in kinetic energy can then be written as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m v_z^2 + \frac{1}{2}m v_\perp^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m v_z^2 + \mu B\right)\ . \tag{3.37}$$

Combining (3.36) and (3.37) one obtains

$$-\mu\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mu B) = 0 \tag{3.38}$$

and finally

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0\,. \tag{3.39}$$

Hence, the magnetic moment is conserved to the same degree of accuracy as the diamagnetic force gave a sufficiently accurate description of the motion of the guiding center.

3.4.2 *The Mirror Effect*

The adiabatic invariance of the magnetic moment can be used to calculate the confinement properties of a magnetic mirror field. A natural magnetic mirror is given by the Earth's magnetic dipole field. Following a field line from the magnetic equator towards the pole, one observes the increase in field line density and hence in magnetic flux density (see Fig. 3.10). Therefore, charged particles can be trapped between the mirrors at North and South pole.

Magnetic mirrors can also be formed in the laboratory, e.g., between a set of circular magnetic field coils, as shown in Fig. 3.10b. When the distance of the coils is larger than their radius, the magnetic field becomes inhomogeneous along the axis of the system, as can be seen by the high density of field lines at the coil position and the lower density in the midplane. We will now discuss the confinement of a charged particle that moves along the central field line of this system.

At any position, the velocity components of the particle are v_z along the axis and $v_\perp = (v_x^2 + v_y^2)^{1/2}$, which is the speed of gyromotion. In the symmetry plane of the mirror the magnetic field is $B = B_0$ and there the particle has initial velocities $v_{\perp 0}$

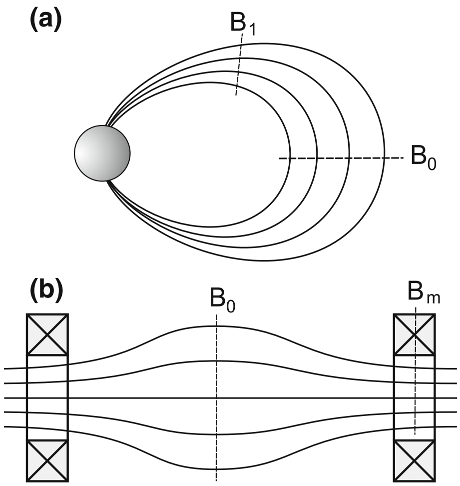

Fig. 3.10 **a** Mirror action of the Earth's dipole field. Following a field line from the magnetic equator to the pole the magnetic field increases $B_1 > B_0$. **b** Magnetic mirror created in the laboratory by a set of field coils

and v_{z0}. The maximum magnetic field $B = B_m$ is found in the vicinity of the mirror coils. The motion of the particle is governed by the conservation of energy and the adiabatic invariance of the magnetic moment, which can be written as

$$\begin{aligned} v_\perp^2 + v_z^2 &= v_{\perp 0}^2 + v_{z0}^2 = v_0^2 \\ v_\perp^2/B &= v_{\perp 0}^2/B_0 \,. \end{aligned} \tag{3.40}$$

When the particle moves into regions of higher magnetic field, its parallel energy is consumed by the diamagnetic force. At the same time, the gyrofrequency increases, which leads to a larger kinetic energy of the gyromotion. A reflection by the magnetic mirror occurs when the energy of parallel motion becomes zero at any position before B attains its maximum value B_m. Solving (3.40) for $v_z = 0$ and setting $v_{\perp 0} = v_0 \sin\theta$ gives

$$0 = v_z^2 = v_0^2 \left(1 - \frac{B}{B_0} \sin^2(\theta)\right) \,. \tag{3.41}$$

This means that the stopping point of a particle is only dependent on the starting angle θ with respect to the magnetic field. It is independent of the magnitude of the initial velocity v_0. All particles with a starting angle $\theta > \theta_m$ are confined, while the particles with $\theta < \theta_m$ can overcome the mirror point B_m and form the *loss cone* in velocity space. The angle of the loss cone is

$$\theta_m = \arcsin\left(\sqrt{B_0/B_m}\right) \,. \tag{3.42}$$

The quantity $R_m = B_m/B_0$ is called the *mirror ratio*, which defines the confinement quality of a mirror machine. A large mirror ratio is equivalent to a small loss-cone angle.

In Fig. 3.11 a mirror field is shown that is produced by a pair of circular currents, each of magnitude I, at positions $z = \pm L/2$ and radius R. From Biot and Savart's law, the magnetic field on the axis of a current ring is

$$B(z) = \frac{\mu_0}{2} \frac{I R^2}{(R^2 + z^2)^{3/2}} \,. \tag{3.43}$$

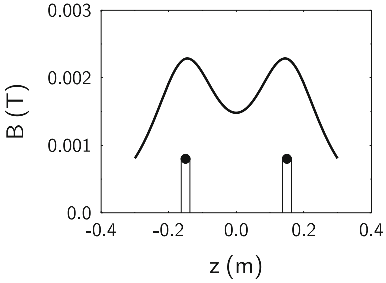

Fig. 3.11 Mirror field of an arrangement of two ring currents. The positions and radius of the ring currents are indicated

Hence the total mirror field becomes $B_{\text{tot}} = B(z - L/2) + B(z + L/2)$. For $R = 0.15\,\text{m}$, $L = 0.3\,\text{m}$, $I = 500\,\text{A}$ the mirror ratio becomes $B_{\text{m}}/B_0 = 1.48$ and the half-angle of the loss-cone $\theta = 55.3°$.

3.4.3 The Longitudinal and the Flux Invariant

According to the three degrees of freedom of motion, we can define exactly three adiabatic invariants. The first is the magnetic moment, which corresponds to the periodic gyromotion. In a magnetic mirror, trapped particles are bouncing back and forth, on a slower time scale, between the reflection points, which can be seen in Fig. 3.12. There, an energetic proton is trapped in the dipole field of the Earth. With this secondary periodic moment, we associate the *second adiabatic invariant* J, also called the *longitudinal invariant*,

$$J = \int v_\parallel \text{d}l \; . \tag{3.44}$$

The integral is taken between the reflection points. The invariant J is more fragile than the fairly robust magnetic moment μ.

The third periodic motion, on an even longer time scale, is associated with the toroidal drift of this bouncing trajectory in the curved dipole field (see Fig. 3.12), which leads to a circular motion in the equatorial plane. Associated with this slow periodic drift motion is the third or *flux invariant* Φ, which represents the total magnetic flux encircled by the drifting bounce-trajectory. Φ is an even more fragile quantity than J, and is rarely used for calculations.

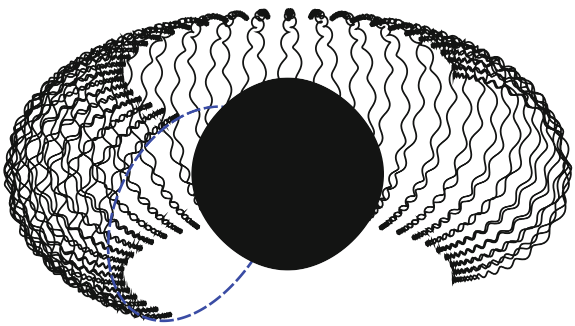

Fig. 3.12 Mirror effect and particle drifts in the Earth's dipole field. A 10 MeV proton with a pitch angle of $\theta = 30°$ starting at $3R_E$ is trapped in the Earth magnetic field. The initial field line, on which the particle motion started, is shown by the *dashed curve*. The particle performs a hierarchy of three periodic motions: gyration about the field line, bouncing between the mirror points, and a slow (toroidal) drift in the equatorial plane