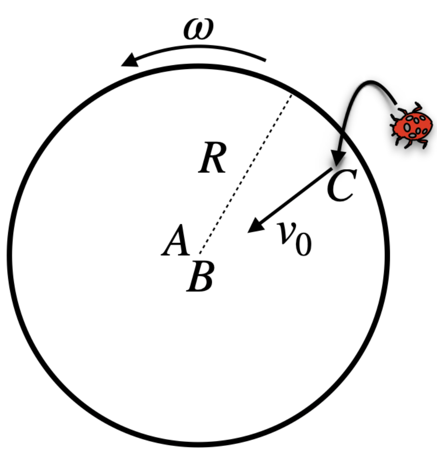
HW 4

1 Ladybug on a Carousel

A horizontal carousel with radius R rotates with angular velocity ω. Observer A stands at the center of the carousel at rest (without rotating). Observer B stands at the center of the carousel and rotates with it. Observer C stands at the edge of the carousel and rotates with it. A ladybug sits near the carousel at rest, jumps on the carousel the moment observer C passes near it, and moves towards the center with constant velocity v_0 relative to the carousel. Find the position, velocity and acceleration of the ladybug as follows: (note: use each observer as the origin of his coordinate system)

1. Relative to observer C, using polar coordinates $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\right)$ pointing to the ladybug.
2. Relative to observer B, using polar coordinates $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\right)$ pointing to the ladybug.
3. Relative to observer B, using Cartesian coordinates $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$, static relative to observer B (i.e. rotate with the carousel). Assume that the ladybug jumped on the carousel at angle θ relative to the Cartesian coordinate system of observer B.
4. Relative to observer A, using polar coordinates $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}\right)$ pointing to the ladybug.
5. Relative to observer A, using static Cartesian coordinates $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. What does the trajectory looks like?



Solution:

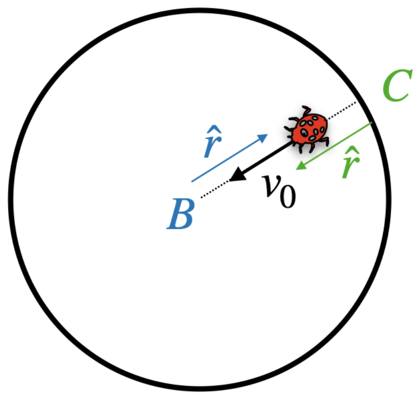
1. The ladybug's movement relative to observer C is straightforward, since they are both rotating with the carousel, the origin is the initial point of the ladybug and it moves along the same direction relative to the carousel (towards the center). Therefore the position of the ladybug is simply $v_0 t$ in the direction

of $\hat{\boldsymbol{r}}$ (originated at observer C) with a constant angle θ_0 corresponding to the chosen orientation of the coordinate system, which can be chosen to be 0. Hence,

$$\begin{aligned} \boldsymbol{r} &= v_0 t \hat{\boldsymbol{r}}, \\ \boldsymbol{v} = \dot{\boldsymbol{r}} &= v_0 \hat{\boldsymbol{r}} \text{ (since } \dot{\theta} = 0), \\ \boldsymbol{a} = \dot{\boldsymbol{v}} &= 0. \end{aligned}$$

2. Observer B experiences a similar movement as observer C does, only with a different origin of his coordinate system. He sees the ladybug moves towards him along the same direction (since both rotate with the carousel).Therefore

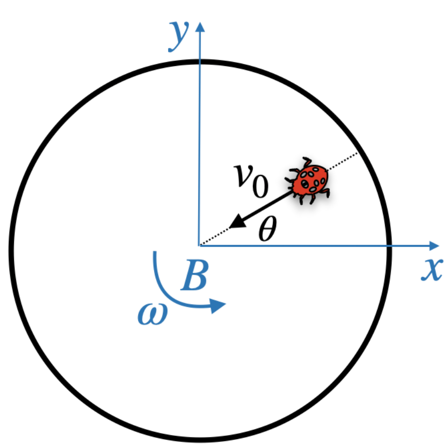
$$\begin{aligned} \boldsymbol{r} &= (R - v_0 t)\, \hat{\boldsymbol{r}}, \\ \boldsymbol{v} = \dot{\boldsymbol{r}} &= -v_0 \hat{\boldsymbol{r}} \text{ (since } \dot{\theta} = 0), \\ \boldsymbol{a} = \dot{\boldsymbol{v}} &= 0. \end{aligned}$$



3. This is the same movement described in (2) only represented by Cartesian coordinates. Using the transformations rules from polar coordinates to Cartesian we find

$$\hat{\boldsymbol{r}} = \cos\theta\, \hat{\boldsymbol{x}} + \sin\theta\, \hat{\boldsymbol{y}} \quad \rightarrow \quad \begin{cases} \boldsymbol{r} = (R - v_0 t)\,(\cos\theta\, \hat{\boldsymbol{x}} + \sin\theta\, \hat{\boldsymbol{y}}) \\ \boldsymbol{v} = -v_0\,(\cos\theta\, \hat{\boldsymbol{x}} + \sin\theta\, \hat{\boldsymbol{y}})\,, \end{cases}$$

and of course that $\boldsymbol{a} = 0$.



4. Observer A see the ladybug almost as observer B, only rotating (since he is not rotating with the carousel). Therefore

$$\boldsymbol{r} = (R - v_0 t)\, \hat{\boldsymbol{r}},$$

only this time,

$$\begin{aligned}\boldsymbol{v} &= \dot{\boldsymbol{r}} \\ &\text{(since now } \dot{\theta} \neq 0\text{)} \\ &= -v_0 \hat{\boldsymbol{r}} + (R - v_0 t)\, \omega \hat{\boldsymbol{\theta}},\end{aligned}$$

also

$$\begin{aligned}\boldsymbol{a} &= \dot{\boldsymbol{v}} \\ &= -v_0 \frac{d}{dt}\hat{\boldsymbol{r}} - v_0 \omega \hat{\boldsymbol{\theta}} + (R - v_0 t)\, \omega \frac{d}{dt}\hat{\boldsymbol{\theta}} \\ &= -v_0 \omega \hat{\boldsymbol{\theta}} - v_0 \omega \hat{\boldsymbol{\theta}} - (R - v_0 t)\, \omega^2 \hat{\boldsymbol{r}} \\ &= -(R - v_0 t)\, \omega^2 \hat{\boldsymbol{r}} - 2 v_0 \omega \hat{\boldsymbol{\theta}}.\end{aligned}$$

5. Again, all we need to do is use the result in (4) and the transformation rules,

$$\begin{aligned}\hat{\boldsymbol{r}} &= \cos\theta\, \hat{\boldsymbol{x}} + \sin\theta\, \hat{\boldsymbol{y}}, \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\, \hat{\boldsymbol{x}} + \cos\theta\, \hat{\boldsymbol{y}}.\end{aligned}$$

Using $\theta = \omega t$, we find

$$\boldsymbol{r} = (R - v_0 t)\left(\cos\omega t\, \hat{\boldsymbol{x}} + \sin\omega t\, \hat{\boldsymbol{y}}\right),$$

while the velocity is

$$\begin{aligned}\boldsymbol{v} &= -v_0 \hat{\boldsymbol{r}} + (R - v_0 t)\, \omega \hat{\boldsymbol{\theta}} \\ &= -v_0 \left(\cos\omega t\, \hat{\boldsymbol{x}} + \sin\omega t\, \hat{\boldsymbol{y}}\right) + (R - v_0 t)\, \omega \left(-\sin\omega t\, \hat{\boldsymbol{x}} + \cos\omega t\, \hat{\boldsymbol{y}}\right) \\ &= -\left[(R - v_0 t)\, \omega \sin\omega t + v_0 \cos\omega t\right] \hat{\boldsymbol{x}} + \left[(R - v_0 t)\, \omega \cos\omega t - v_0 \sin\omega t\right] \hat{\boldsymbol{y}}.\end{aligned}$$

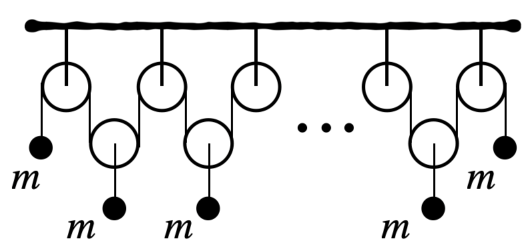
The acceleration expression reads

$$\begin{aligned}\boldsymbol{a} &= -(R - v_0 t)\, \omega^2 \left(\cos\omega t\, \hat{\boldsymbol{x}} + \sin\omega t\, \hat{\boldsymbol{y}}\right) - 2 v_0 \omega \left(-\sin\omega t\, \hat{\boldsymbol{x}} + \cos\omega t\, \hat{\boldsymbol{y}}\right) \\ &= \left[2 v_0 \omega \sin\omega t - (R - v_0 t)\, \omega^2 \cos\omega t\right] \hat{\boldsymbol{x}} - \omega \left[(R - v_0 t)\, \omega \sin\omega t\, \hat{\boldsymbol{y}} + 2 v_0 \cos\omega t\right] \hat{\boldsymbol{y}}.\end{aligned}$$

The trajectory looks like either a spiral or a curve going inwards, since the radial motion indicates a decreasing radius $\rho(t) = R - v_0 t$, while the angular motion is with constant angular velocity ω. The trajectory's shape will depend on the ratio between the velocity of the ladybug and the angular velocity of the carousel.

2 $N + 2$ Masses on Pulleys

$N + 2$ masses are hanged from a system of pulleys as shown in the figure. What is the acceleration of the masses at the edges of the system?



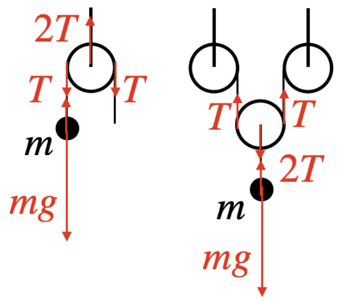
Solution:

Let us write Newton's II law for a mass at the edge and at the middle of the system

$$\text{edge: } T - mg = ma$$
$$\text{middle: } 2T - mg = ma',$$

hence

$$-g = 2a - a'.$$



In order to get a relation between the accelerations a and a' first let us consider 1 mass in the middle instead of N. When the masses at the edges shift, say Δy each, they give or take $2\Delta y$ length of rope. The mass in the middle shifts the other direction with a factor of $1/2$, since it takes twice the length to lower its pulley, i.e. $\Delta y' = -2\Delta y/2 = -\Delta y$. Considering N such masses in the middle we must divide the length of the rope equally between them, thus use a factor of $1/2N$. In order to get a condition for the acceleration we take the time derivatives

$$\Delta y' = -\Delta y/N$$
$$v' = -v/N$$
$$a' = -a/N.$$

Plugging the accelerations relation into the equation we've found earlier yields

$$a = -\frac{g}{2 + \frac{1}{N}},$$

using this results we also find the acceleration of the masses in the middle to be

$$a' = \frac{g}{2N + 1}.$$

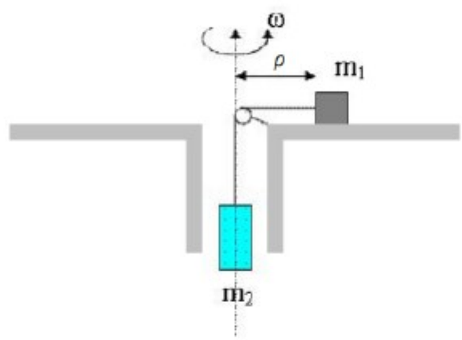
We should check ourselves by taking the limit of $N = 0$, we find that $a \to 0$ which makes sense because then we have only two masses balancing each other. Another interesting limit is that of $N \to \infty$, for which we find a constant acceleration for the masses at the edges, $a = -g/2$, while $a' \to 0$. A somewhat disturbing result, which rises due to the infinite zero-like shifts of the rope.

3 Turning Table (Midterm 2018)

Mass m_1 is at rest on a horizontal table which is turning with an angular velocity ω around its center (details in the figure). Mass $m_1$on the table is connected to a mass m_2 hanging beneath the center of the table by a string of negligible mass through a frictionless pulley.
There is a frictional force f between the table and mass $m_1$that will act in the required direction to prevent the mass from moving.
The magnitude of this force is limited by:

$$|f| < 3N$$

Where N is the force that the table exerts on the mass m_1.



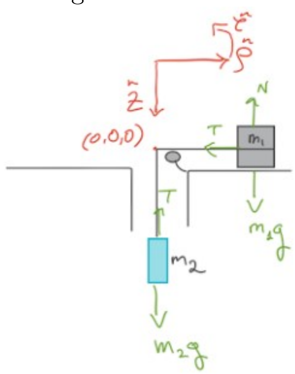
a. What is the lenght range of ρ ,the length of the horizontal part of the connecting string, for which the masses are at rest relative to the table?

b. For some ρ_0 in the range you found in (**a.**), show that the minimal angular velocity of which m_1 will start moving outward is given by

$$\omega_0 = \sqrt{(m_2 + 3m_1)g/(m_1\rho_0)}.$$

Solution:

a. Let us set a polar coordinate system for m_1 with a $\hat{z}$ axis directed downwards.
The origin is set to be at center of the table. Looking at the force diagram below:



The equation of motion for m_2 in the $\hat{z}$ direction:

$$m_2 g - T = m_2 a \Rightarrow \{\text{static picture, a=0}\} \Rightarrow T = m_2 g$$

For m_1 in the $\hat{z}$ direction:

$$m_1 g - N = m_1 a_z \Rightarrow \{\text{the mass stays on the table, } a_z = 0\} \Rightarrow N = m_1 g$$

For m_1 in the radial direction:

$$-T = m_1 a_r \Rightarrow \{\text{static picture, but we have centrifugal acceleration}\}$$

$$-T + f = -m_1 \omega^2 \rho$$
$$f = T - m_1 \omega^2 \rho$$

So the condition is:

$$-3N < T - m_1 \omega^2 \rho < 3N$$

Setting in $N = m_1 g$ and $T = m_2 g$ we get:

$$-3m_1 g < m_2 g - m_1 \omega^2 \rho < 3m_1 g$$

Which become:

$$\frac{g\,(m_2 - 3m_1)}{m_1 \omega^2} < \rho < \frac{g\,(m_2 + 3m_1)}{m_1 \omega^2}$$

b. The moment before the table is turning fast enough for mass $m_1$to move outwards, we got a frictional force f opposing the motion and therefore directed inwards.
So all the calcuations are the same as section **a** but now we know the sign of the frictional force.
We can write the equation of motion for $m_1$in the radial direction:

$$-T - f = -m_1 \omega^2 \rho$$

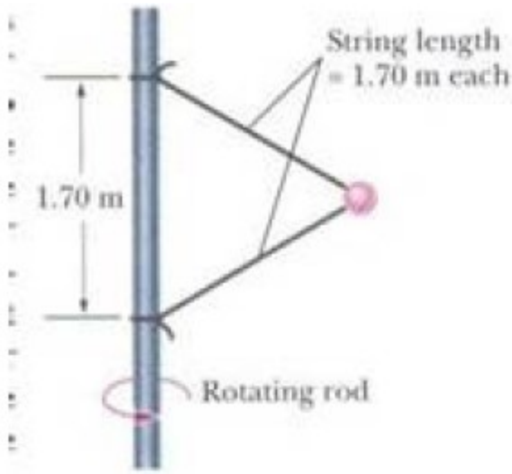
For the minimal ω we set the boundary of f and we get:

$$-T - 3N = -m_2 g - 3m_1 g = -m_1 \omega_0^2 \rho_0$$

$$\omega_0 = \sqrt{\frac{(m_2 + 3m_1)g}{m_1 \rho_0}}$$

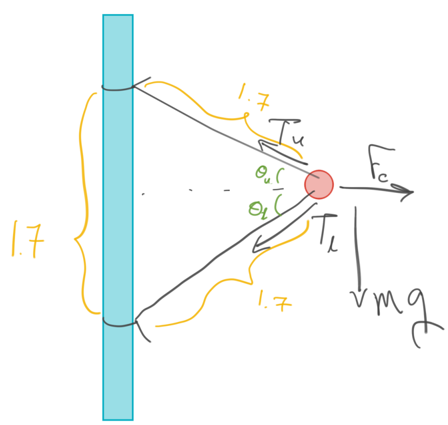
4 Ball With Two Strings

A ball of mass $m = 1.4\,[kg]$ is connected by two strings to a rod rotating around its axis.
The tension in the upper string is $35\,[N]$, both strings are of length $1.7\,[m]$ and are stretched.
The distance between the tying points of the strings to the rod is $1.7\,[m]$ as well.
1. Find the tension in the lower string.
2. Find the total force acting on the ball.
3. Find the velocity of the ball (radial and tangential).



Solution:

Let us mark the angle between the upper string and the horizon as θ_u and the angle between the lower string and the horizon as θ_l, as shown in the figure:



Let's also define a coordinate system that rotating with the rod with y axis directed up and x axis directed to the right.
The forces that acts on the ball are:

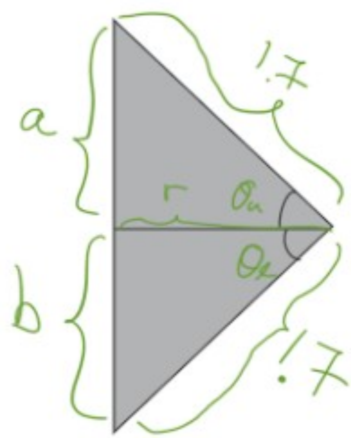
- Upper string tension:
$$\boldsymbol{T_u} = T_u\left(-\cos\theta_u\hat{x} + \sin\theta_u\hat{y}\right)$$
- Lower string tension:
$$\boldsymbol{T_l} = T_l\left(-\cos\theta_l\hat{x} - \sin\theta_l\hat{y}\right)$$
- Centrifugal force:
$$\boldsymbol{F_c} = m\frac{v^2}{r}\hat{x}$$

where, v is the tangential velocity of the ball and r is its distance from the rod.

- Gravitational force:

$$\boldsymbol{F_g} = -mg\hat{y}$$

Now let's use geometry:



And we can find that $\theta_u = \theta_l := \theta \Rightarrow a = b = \frac{1.7\,[m]}{2} = 0.85\,[m]$ and that

$$\sin\theta = \frac{0.85}{1.7} = \frac{1}{2}$$

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \frac{\sqrt{3}}{2}$$

And becuase $\cos\theta = \frac{r}{1.7} \Rightarrow r = 1.47\,[m]$.
Nothing is moving relative to the rotating rod frame of reference so the radial velocity of the ball is zero and $\boldsymbol{F} = 0$.
Using newton's second law we get:

$$\begin{aligned} -\cos\theta\,(T_u + T_l) + m\tfrac{v^2}{r} = 0 \quad & \hat{\boldsymbol{x}} \\ \sin\theta\,(T_u - T_l) - mg = 0 \quad & \hat{\boldsymbol{y}} \end{aligned}$$

Solving the equation for $\hat{\boldsymbol{y}}$ we get:

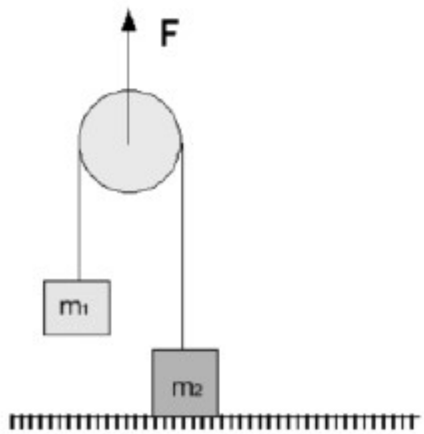
$$T_l = T_u - \frac{mg}{\sin\theta} = \boldsymbol{7\,[N]}$$

Solving the equation for $\hat{\boldsymbol{x}}$ we get:

$$v^2 = \frac{r\cos\theta}{m}\,(T_u + T_l) = 38.2\,\left[\frac{m^2}{s^2}\right]$$

$v = 6.2\frac{m}{s}$ is the tangential velocity.

5 Rising pulley



A force is exerted directly up on the axis of a pulley.
Consider the pulley and string to be massless and the pulley has no friction.
Two objects: m_1 with mass $1.2\,kg$ and m_2 with mass $1.9\,kg$ are attached to the opposite ends if the string which passes over the pulley.
The object m_2 is in contact with the floor.
a) What is the largest value of the force $\overrightarrow{F}$ may have so that $m_2$will remain on the floor?
b) What is the tension in the string if the upward force F is $110\,N$?
c) With the tension found in part (b), what is the acceleration of m_1?

Solution:

a)
To keep m_2 on the floor its acceleration need to be equal to zero $\ddot{y_2} = 0$.
The contact with the floor is described by the normal force N and m_2 is touching the floor if $N \geq 0$.
Writting the equation of motion:

$$\begin{array}{cc} \boldsymbol{Pulley:} & F - 2T = m_p\ddot{y}_p = 0 \\ \boldsymbol{m_1}: & T - m_1 g = m_1\ddot{y}_1 \\ \boldsymbol{m_2}: & T - m_2 g + N = m_2\ddot{y}_2 = 0 \end{array}$$

So we get

$$F = 2T$$

from the puuley's equation. And m_2 eqn gives

$$N = m_2 g - T = m_2 g - \frac{F}{2} \geq 0$$

$$F \leq 2m_2 g$$

Then the maximal value of the force will be $F_{MAX} = 38\,N$.
b)Setting $F = 110\,N$, $T = \frac{F}{2} \Rightarrow$

$$T = 55\,N$$

c)Solving the equation of motion for $m_1 \Rightarrow$

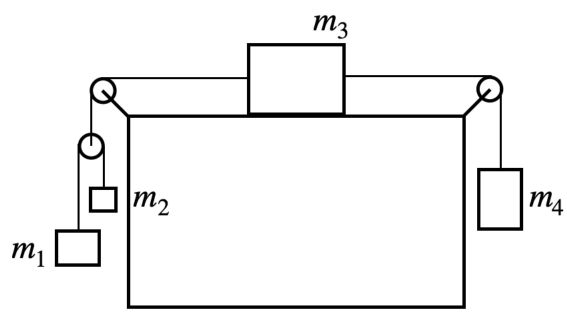
$$T - m_1 g = m_1\ddot{y}_1 \Rightarrow \ddot{y}_1 = \frac{T - m_1 g}{m_1}$$

And we can find that m_1's acceleration is $36\,\frac{m}{s^2}$.

6 Masses with Pulleys

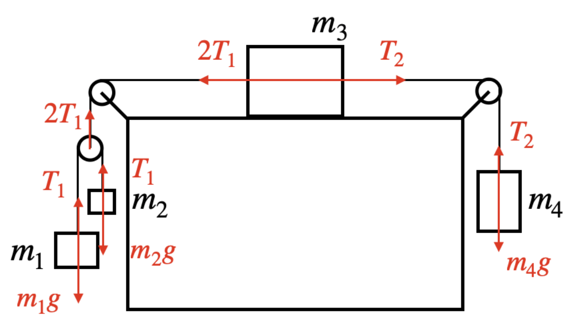
A system of 4 masses, connected by pulleys as shown in the figure, is released from rest. Given that there is no friction and that m_1 remains static:

1. Express m_4 using m_1, m_2 and m_3.
2. Find the maximal value for the ratio m_1/m_2.



Solution:

Let us begin by writing all the relevant forces:



The vertical forces on m_3 are irrelevant since is no vertical movement or friction. Next we write the equations for each mass:

$$\begin{aligned}
m_1 &: \ T_1 - m_1 g = 0, \\
m_2 &: \ T_1 - m_2 g = m_2 a_2, \\
m_3 &: \ T_2 - 2T_1 = m_3 a_3, \\
m_4 &: \ T_2 - m_4 g = m_4 a_4.
\end{aligned}$$

Next we write the relations between the accelerations a_2, a_3 and a_4:

$$a \equiv a_3 = -a_4 = \frac{1}{2} a_2,$$

since a_3 and a_4 are connected by the same rope (the minus sign is due to definitions of directions), while m_2 must shift twice the distance m_3 and m_4 shift in order to keep m_1 static. Reducing the number of equations by using the equation for m_1 and m_4 we find

$$\begin{aligned}
m_2 &: m_1 g - m_2 g = 2 m_2 a, \\
m_3 &: m_4 (g - a) - 2 m_1 g = m_3 a.
\end{aligned}$$

1. Getting rid of a yields the equation

$$m_4 - 2m_1 = (m_3 + m_4)\,\frac{m_1 - m_2}{2m_2},$$

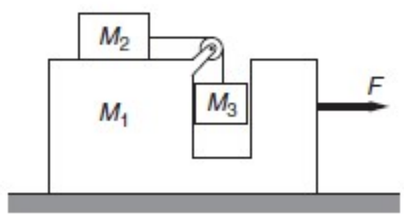
   which is followed by the relation

$$m_4\left(\frac{3m_2 - m_1}{2m_2}\right) = \frac{m_1m_3 - m_2m_3 + 4m_1m_2}{2m_2} \quad\rightarrow\quad m_4 = \frac{m_1m_3 - m_2m_3 + 4m_1m_2}{3m_2 - m_1}.$$

2. We immediately see that in the denominator of the expression for m_4 there is a singularity if $3m_2 = m_1$. This means that if $m_1 \rightarrow 3m_2$ then even $m_4 \rightarrow \infty$ (which corresponds to a free fall of m_4, with acceleration g) cannot result in m_1 remain static.

7 Pedagogical Machine

A "pedagogical machine" is illustrated in the sketch. All surfaces are frictionless.
1.What force $\overrightarrow{\boldsymbol{F}}$ must be applied to M_1 to keep M_3 from rising or falling?
2.Now a different force $\overrightarrow{\boldsymbol{F}}$ is pulling M_1(different than the $\overrightarrow{\boldsymbol{F}}$ you found in section 1).
What is the acceleration of M_1?
3.When there is no force. What will be the accelerations of M_2 and M_3?
4.Is there a force $\overrightarrow{\boldsymbol{F}}$ that will act on M_2 that will cause the whole system to move with a constant velocity? If this force exist, what is it?



Solution:

1.Writting the equation of motion -

$$\boldsymbol{M_1, x} \Rightarrow F - N_{13} = M_1\ddot{x_1}$$
$$\boldsymbol{M_2, x} \Rightarrow T = M_2\ddot{x_2}$$
$$\boldsymbol{M_3, y} \Rightarrow T - M_3g = M_3\ddot{y_3}$$
$$\boldsymbol{M_3, x} \Rightarrow N_{13} = M_3\ddot{x_3}$$

To demand that M_3 won't fall or rise we set $\ddot{y_3} = 0$ so $T = M_3g$.
Since M_3 touching M_1 they have to move with equal accelerations $\ddot{x_1} = \ddot{x_3}$ and we get that $N_{13} = M_3\ddot{x_1}$.
The equation for M_1 yields $F = (M_1 + M_3)\,\ddot{x}_1$.
The tensed cable has a fixed length, so we have a constraint

$$(x_2' - x_p) + (y_p - y_3) = l \text{ (const)}$$

where x_2' is the position of M_2 relative to M_1 so $x_2' = x_2 - x_1$.
Taking the second derivative we get

$$\ddot{x}_2 - \ddot{x}_1 = \ddot{y}_3 = 0$$

So $\ddot{x}_2 = \ddot{x}_1 = \frac{T}{M_2}$ where the second equality is from the M_2 eqn.

And

$$F = (M_1 + M_3)\,\frac{M_3}{M_2} g$$

2.
Now $\ddot{y}_3 \neq 0$ and $\ddot{x}_1 = \ddot{x}_3$, $N_{13} = M_3\ddot{x}_1$and $F = (M_1 + M_3)\,\ddot{x}_1$ still holds.
So

$$\ddot{x}_1 = \frac{F}{M_1 + M_3}$$

3. If $F = 0$ then $\ddot{x}_1 = \ddot{x}_3 = 0$.
From the cable constraint we get $\ddot{x}_2 = \ddot{y}_3 \equiv a$.

$$M_2 a - M_3 g = M_3 a$$

and M_2 and M_3 accelerations equal to

$$a = \frac{M_3}{M_2 - M_3} g$$

4. It is impossible to apply a force on the system and have it move at constant velocity.